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Estimates on Solutions of a New Quadruple Integral Inequalities

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Abstract: With the development of science and technology and the progress of society, differential equations have played important roles in natural science, social science, engineering technology, etc. By investigating the properties of differential equation solutions, some phenomena could be explained, future development trends could be predicted, and reference and theoretical bases for decision-making could be provided. However, in several cases, explicit solutions cannot be derived for differential equations, but using appropriate integral inequality techniques, one could explore the existence, uniqueness, vibration, stability and other qualitative properties of differential equation solutions and estimate solution sizes of differential equations. Therefore, researchers are investigating integral inequalities and constantly enriching the existing achievements. Based on previous research works, a class of four-fold nonlinear integral inequalities with unknown derivative functions was established in this research. By using various inequality analysis methods such as variable substitution, amplification, differentiation and integration, estimations were provided for unknown functions in quadruple integral inequalities. This type of inequality could be applied to evaluate the estimation of corresponding differential equation solutions, so as to provide an effective mathematical tool to solve practical problems.

Keywords: Integral Inequality; Quadruple Integral; Differential Equation; Estimation

1. Introduction

Differential equations have been extensively applied in almost all fields of natural science. For instance, in physics, differential equations are being applied to describe objects movements, vibrations and other phenomena. In chemistry, it is employed to describe the kinetic process of chemical reactions, in engineering, it is used to describe the dynamic behaviors of various systems. In economics, it is employed to develop different economic models, predict market trends, formulate economic policies, etc. However, investigation on integral inequalities provides a fruitful theoretical tool to solve differential equations.

Gronwall [1] developed the following integral inequalities to evaluate the continuous dependence of differential equation solutions on the parameters.

$$u(t) \le c + \int_a^t f(s)u(s)ds \qquad (t \in [a,b]) \tag{1}$$

Where constant c ≥ 0. The estimated solution for the unknown function was

$$u(t) \le c\exp\left(\int_a^t f(s)ds\right) \qquad (t \in [a,b]) \tag{2}$$

Later, it was seen that Gronwalltype integral inequality was an important tool to explore differential and integral equations. Therefore, researchers continue to investigate and popularize it, so that its application range in differential and integral equations continues to increase [2-5].

In 1998, Pachpatte [6] investigated the integral inequality involving unknown functions within the integral sign and their derivatives. In 2014, Zareen [7] further explored nonlinear integral inequalities. In 2019, Huang and Wang et al. [8-9] investigated a class of nonlinear double integral inequalities containing unknown functions within the integral sign and their derivatives as well as a class of triple integral inequalities containing unknown derivations. In 2022, Lu and Huang [10] extended the results of reference [8] to triple integration.

Based on the above research results, this research constructed the following four-fold nonlinear integral inequalities with unknown functions within the integral sign and their derivatives:

$$\begin{aligned} u'(\omega) \le{}& p(\omega) + q(\omega)\{u(\omega) + \int_{\omega_0}^{\omega}[a(\xi)(u(\xi)+u'(\xi)) + a(\xi)\int_{\omega_0}^{\xi}[c(\sigma)(u(\sigma) \\ &+ u'(\sigma)) + c(\sigma)\int_{\omega_0}^{\sigma}[d(\eta)((u(\eta)+u'(\eta)) + d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)u'(\varphi) \\ &\times (u(\varphi)+u'(\varphi)) + g(\varphi)]d\varphi]d\eta]d\sigma]d\xi\} \\ & \omega \in [\omega_0, \infty) \end{aligned} \tag{3}$$

Estimation of unknown function in quadruple integral inequality (3) was performed using various inequality analysis methods such as variable substitution, amplification, differentiation and integration.

2. Main Result

In order to simplify the proof of the results, the following Lemma was developed.

Lemma 1. [8] Let $a(\omega)$, $b(\omega)$, $c(\omega)$ be nonnegative, continuous and given functions on $[\omega_0,\infty)$, respectively, and $a(\omega)$ be an increasing function on $[\omega_0,\infty)$. Then, $u(\omega)$ would be unknown function satisfying the following inequation.

$$u(\omega)\le a(\omega)+\int_{\omega_0}^{\omega}b(\upsilon)u(\upsilon)d\upsilon+\int_{\omega_0}^{\omega}c(\upsilon)u^2(\upsilon)d\upsilon \quad (\omega\in[\omega_0,\infty)) \tag{4}$$

If $\exp(-\ln a(\omega))-\int_{\omega_0}^{\omega}b(\upsilon)d\upsilon)-\int_{\omega_0}^{\omega}c(\upsilon)d\upsilon>0,$ then

$$u(\omega)\le(\exp(-\ln a(\omega))-\int_{\omega_0}^{\omega}b(\upsilon)d\upsilon)-\int_{\omega_0}^{\omega}c(\upsilon)d\upsilon)^{-1} \quad (\omega\in[\omega_0,\infty)) \tag{5}$$

Theorem 1. Suppose $p(\omega),q(\omega),a(\omega),c(\omega),d(\omega),f(\omega),g(\omega)$ to be nonnegative and continuous and given functions on $[\omega_0,\infty)$. The unknown functions $u(\omega)$ and $u'(\omega)$ are defined on $[\omega_0,\infty)$. Then, $u(\omega)$ would be unknown function satisfying (3).

If

$$\exp(-\ln(u(\omega_0))+\int_{\omega_0}^{\omega}A(\upsilon)d\upsilon)-\int_{\omega_0}^{\omega}B(\upsilon)d\upsilon)-\int_{\omega_0}^{\omega}C(\upsilon)d\upsilon>0 \quad (\omega\in[\omega_0,\infty)) \tag{6}$$

then

$$\begin{aligned}u(\omega)\le u(\omega_0)+\int_{\omega_0}^{\omega}\{p(\upsilon)[1+a(\upsilon)+c(\upsilon)+d(\upsilon)]+[q(\upsilon)(1+a(\upsilon)+c(\upsilon)+d(\upsilon))\\+a(\upsilon)+c(\upsilon)+d(\upsilon)]Z(\upsilon)\}d\upsilon \quad (\omega\in[\omega_0,\infty))\end{aligned} \tag{7}$$

where

$$Z(\omega):=(\exp(-\ln(u(\omega_0))+\int_{\omega_0}^{\omega}A(\upsilon)d\upsilon)-\int_{\omega_0}^{\omega}B(\upsilon)d\upsilon)-\int_{\omega_0}^{\omega}C(\upsilon)d\upsilon)^{-1} \tag{8}$$

$$A(\omega):=f(\omega)p^2(\omega)+[1+a(\omega)+c(\omega)+d(\omega)]p(\omega)+g(\omega) \tag{9}$$

$$\begin{aligned}B(\omega):=q(\omega)+a(\omega)+a(\omega)q(\omega)+c(\omega)+c(\omega)q(\omega)\\+d(\omega)+d(\omega)q(\omega)+f(\omega)p(\omega)+2f(\omega)p(\omega)q(\omega)\end{aligned} \tag{10}$$

$$C(\omega):=f(\omega)q(\omega)+f(\omega)q^2(\omega) \tag{11}$$

Proof. Let

$$\begin{aligned}z_1(\omega)=u(\omega)+\int_{\omega_0}^{\omega}[a(\xi)(u(\xi)+u'(\xi))+a(\xi)\int_{\omega_0}^{\xi}[c(\sigma)(u(\sigma)+u'(\sigma))\\+c(\sigma)\int_{\omega_0}^{\sigma}[d(\eta)((u(\eta)+u'(\eta))+d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)u'(\varphi)\\\times(u(\varphi)+u'(\varphi))+g(\varphi)]d\varphi]d\eta]d\sigma]d\xi\\\omega\in[\omega_0,\infty)\end{aligned} \tag{12}$$

From (3) and (12), we have $z_1(\omega_0)=u(\omega_0)$, $u(\omega)\le z_1(\omega)$

$$u'(\omega)\le p(\omega)+q(\omega)z_1(\omega) \tag{13}$$

Differentiating $z_1(\omega)$ with respect to ω, using (3) gives

$$\begin{aligned}z_1'(\omega)&=u'(\omega)+a(\omega)[u(\omega)+u'(\omega)]+a(\omega)\int_{\omega_0}^{\omega}[c(\sigma)(u(\sigma)+u'(\sigma))\\&\quad+c(\sigma)\int_{\omega_0}^{\sigma}[d(\eta)((u(\eta)+u'(\eta))+d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)u'(\varphi)\\&\quad\times(u(\varphi)+u'(\varphi))+g(\varphi)]d\varphi]d\eta]d\sigma\\&\le p(\omega)+q(\omega)z_1(\omega)+a(\omega)[z_1(\omega)+p(\omega)+q(\omega)z_1(\omega)]\\&\quad+a(\omega)\int_{\omega_0}^{\omega}[c(\sigma)[z_1(\sigma)+p(\sigma)+q(\sigma)z_1(\sigma)]\\&\quad+c(\sigma)\int_{\omega_0}^{\sigma}[d(\eta)(z_1(\eta)+p(\eta)+q(\eta)z_1(\eta))\\&\quad+d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)(p(\varphi)+q(\varphi)z_1(\varphi))\\&\quad\times(z_1(\varphi)+p(\varphi)+q(\varphi)z_1(\varphi))+g(\varphi)]d\varphi]d\eta]d\sigma\\&=p(\omega)[1+a(\omega)]+[q(\omega)+a(\omega)+a(\omega)q(\omega)]z_1(\omega)\\&\quad+a(\omega)\int_{\omega_0}^{\omega}[c(\sigma)p(\sigma)+c(\sigma)(1+q(\sigma))z_1(\sigma)\\&\quad+c(\sigma)\int_{\omega_0}^{\sigma}[d(\eta)p(\eta)+d(\eta)(1+q(\eta))z_1(\eta)\\&\quad+d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)p^2(\varphi)+[f(\varphi)p(\varphi)\\&\quad+2f(\varphi)p(\varphi)q(\varphi)]z_1(\varphi)+[f(\varphi)q(\varphi)\\&\quad+f(\varphi)q^2(\varphi)]{z_1}^2(\varphi)+g(\varphi)]d\varphi]d\eta]d\sigma\\&\qquad\omega\in[\omega_0,\infty)\end{aligned} \tag{14}$$

Let

$$\begin{aligned}z_2(\omega)=z_1(\omega)+\int_{\omega_0}^{\omega}[c(\sigma)p(\sigma)+c(\sigma)(1+q(\sigma))z_1(\sigma)\\+c(\sigma)\int_{\omega_0}^{\sigma}[d(\eta)p(\eta)+d(\eta)(1+q(\eta))z_1(\eta)\\+d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)p^2(\varphi)+[f(\varphi)p(\varphi)\\+2f(\varphi)p(\varphi)q(\varphi)]z_1(\varphi)\\+[f(\varphi)q(\varphi)+f(\varphi)q^2(\varphi)]{z_1}^2(\varphi)+g(\varphi)]d\varphi]d\eta]d\sigma\\\omega\in[\omega_0,\infty)\end{aligned} \tag{15}$$

Then $z_2(\omega_0)=z_1(\omega_0)$, $z_1(\omega)\le z_2(\omega) \quad \omega\in[\omega_0,\infty)$ (16)

From (14)-(16), we have

$$\begin{aligned}z_1'(\omega)&\le p(\omega)[1+a(\omega)]+[q(\omega)+a(\omega)q(\omega)]z_1(\omega)+a(\omega)z_2(\omega)\\&\le p(\omega)[1+a(\omega)]+[q(\omega)+a(\omega)+a(\omega)q(\omega)]z_2(\omega)\\&\qquad\omega\in[\omega_0,\infty)\end{aligned} \tag{17}$$

Differentiating $z_2(\omega)$ with respect to ω, using (17) gives

$$\begin{aligned}z_2'(\omega)&=z_1'(\omega)+c(\omega)p(\omega)+c(\omega)(1+q(\omega))z_1(\omega)+c(\omega)\int_{\omega_0}^{\omega}[d(\eta)p(\eta)\\&\quad+d(\eta)(1+q(\eta))z_1(\eta)+d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)p^2(\varphi)\\&\quad+[f(\varphi)p(\varphi)+2f(\varphi)p(\varphi)q(\varphi)]z_1(\varphi)\\&\quad+[f(\varphi)q(\varphi)+f(\varphi)q^2(\varphi)]{z_1}^2(\varphi)+g(\varphi)]d\varphi]d\eta\\&\le p(\omega)[1+a(\omega)]+[q(\omega)+a(\omega)+a(\omega)p(\omega)]z_2(\omega)\\&\quad+c(\omega)p(\omega)+c(\omega)(1+q(\omega))z_2(\omega)\\&\quad+c(\omega)\int_{\omega_0}^{\omega}d(\eta)p(\eta))+d(\eta)(1+q(\eta))z_2(\eta)\\&\quad+d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)p^2(\varphi)+[f(\varphi)p(\varphi)+2f(\varphi)p(\varphi)q(\varphi)]\\&\quad\times z_2(\varphi)+[f(\varphi)q(\varphi)+f(\varphi)q^2(\varphi)]{z_2}^2(\varphi)+g(\varphi)]d\varphi]d\eta\\&=p(\omega)[1+a(\omega)+c(\omega)]+[q(\omega)+a(\omega)+a(\omega)q(\omega)+c(\omega)\\&\quad+c(\omega)q(\omega)]z_2(\omega)+c(\omega)\int_{\omega_0}^{\omega}[d(\eta)p(\eta)+d(\eta)(1+q(\eta))z_2(\eta)\\&\quad+d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)p^2(\varphi)+[f(\varphi)p(\varphi)+2f(\varphi)p(\varphi)q(\varphi)]\\&\quad\times z_2(\varphi)+[f(\varphi)q(\varphi)+f(\varphi)q^2(\varphi)]{z_2}^2(\varphi)+g(\varphi)]d\varphi]d\eta\end{aligned}$$

$$\omega\in[\omega_0,\infty) \tag{18}$$

Let

$$z_3(\omega)=z_2(\omega)+\int_{\omega_0}^{\omega}[d(\eta)p(\eta)+d(\eta)(1+q(\eta))z_2(\eta)$$
$$+d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)p^2(\varphi)+[f(\varphi)p(\varphi)$$
$$+2f(\varphi)p(\varphi)q(\varphi)]z_2(\varphi)+[f(\varphi)q(\varphi)$$
$$+f(\varphi)q^2(\varphi)]z_2{}^2(\varphi)+g(\varphi)]d\varphi]d\eta$$
$$\omega\in[\omega_0,\infty) \tag{19}$$

then $z_3(\omega_0)=z_2(\omega_0)$, $z_2(\omega)\le z_3(\omega)$, $\quad \omega\in[\omega_0,\infty)$ (20)

From (18)-(20), we have

$$z_2'(\omega)\le p(\omega)[1+a(\omega)+c(\omega)]+[q(\omega)+a(\omega)+a(\omega)q(\omega)$$
$$+c(\omega)q(\omega)]z_2(\omega)+c(\omega)z_3(\omega)$$
$$\le p(\omega)[1+a(\omega)+c(\omega)]+[q(\omega)+a(\omega)+a(\omega)q(\omega)$$
$$+c(\omega)+c(\omega)q(\omega)]z_3(\omega)$$
$$\omega\in[\omega_0,\infty) \tag{21}$$

Differentiating $z_3(\omega)$ with respect to ω, using (21) gives

$$z_3'(\omega)=z_2'(\omega)+d(\omega)p(\omega)+d(\omega)(1+q(\omega))z_2(\omega)+d(\omega)\int_{\omega_0}^{\omega}[f(\varphi)p^2(\varphi)$$
$$+[f(\varphi)p(\varphi)+2f(\varphi)p(\varphi)q(\varphi)]z_2(\varphi)$$
$$+[f(\varphi)q(\varphi)+f(\varphi)q^2(\varphi)]z_2{}^2(\varphi)+g(\varphi)]d\varphi$$
$$\le p(\omega)[1+a(\omega)+c(\omega)]+[q(\omega)+a(\omega)+a(\omega)q(\omega)+c(\omega)$$
$$+c(\omega)q(\omega)]z_3(\omega)+d(\omega)p(\omega)+d(\omega)(1+q(\omega))z_3(\omega)$$
$$+d(\omega)\int_{\omega_0}^{\omega}[f(\varphi)p^2(\varphi)+[f(\varphi)p(\varphi)+2f(\varphi)p(\varphi)q(\varphi)]z_3(\varphi)$$
$$+[f(\varphi)q(\varphi)+f(\varphi)q^2(\varphi)]z_3{}^2(\varphi)+g(\varphi)]d\varphi$$
$$=p(\omega)[1+a(\omega)+c(\omega)+d(\omega)]+[q(\omega)+a(\omega)+a(\omega)q(\omega)+c(\omega)$$
$$+c(\omega)q(\omega)+d(\omega)+d(\omega)q(\omega)]z_3(\omega)$$
$$+d(\omega)\int_{\omega_0}^{\omega}[f(\varphi)p^2(\varphi)+[f(\varphi)p(\varphi)+2f(\varphi)p(\varphi)q(\varphi)]z_3(\varphi)$$
$$+[f(\varphi)q(\varphi)+f(\varphi)q^2(\varphi)]z_3{}^2(\varphi)+g(\varphi)]d\varphi$$
$$\omega\in[\omega_0,\infty) \tag{22}$$

Let

$$z_4(\omega)=z_3(\omega)+\int_{\omega_0}^{\omega}[f(\varphi)p^2(\varphi)+[f(\varphi)p(\varphi)$$
$$+2f(\varphi)p(\varphi)q(\varphi)]z_3(\varphi)$$
$$+[f(\varphi)q(\varphi)+f(\varphi)q^2(\varphi)]z_3{}^2(\varphi)+g(\varphi)]d\varphi$$
$$\omega\in[\omega_0,\infty) \tag{23}$$

Then $z_4(\omega_0)=z_3(\omega_0)$, $z_3(\omega)\le z_4(\omega)$, $\quad \omega\in[\omega_0,\infty)$ (24)

From (22)-(24), we have

$$z_3'(\omega)\le p(\omega)[1+a(\omega)+c(\omega)+d(\omega)]+[q(\omega)+a(\omega)+a(\omega)q(\omega)$$
$$+c(\omega)+c(\omega)q(\omega)+d(\omega)q(\omega)]z_3(\omega)+d(\omega)z_4(\omega)$$
$$\le p(\omega)[1+a(\omega)+c(\omega)+d(\omega)]+[q(\omega)+a(\omega)+a(\omega)q(\omega)$$
$$+c(\omega)+c(\omega)q(\omega)+d(\omega)+d(\omega)q(\omega)]z_4(\omega)$$
$$\omega\in[\omega_0,\infty) \tag{25}$$

Differentiating $z_4(\omega)$ with respect to ω, using (25) gives

$$z_4'(\omega)=z_3'(\omega)+f(\omega)p^2(\omega)+[f(\omega)p(\omega)+2f(\omega)p(\omega)q(\omega)]z_3(\omega)$$
$$+[f(\omega)q(\omega)+f(\omega)q^2(\omega)]z_3{}^2(\omega)+g(\omega)$$
$$\le p(\omega)[1+a(\omega)+c(\omega)+d(\omega)]+[q(\omega)+a(\omega)+a(\omega)q(\omega)$$
$$+c(\omega)+c(\omega)q(\omega)+d(\omega)+d(\omega)q(\omega)]z_4(\omega)$$
$$+f(\omega)p^2(\omega)+[f(\omega)p(\omega)+2f(\omega)p(\omega)q(\omega)]z_4(\omega)$$
$$+[f(\omega)q(\omega)+f(\omega)q^2(\omega)]z_4{}^2(\omega)+g(\omega)$$
$$=[f(\omega)p^2(\omega)+[1+a(\omega)+c(\omega)+d(\omega)]p(\omega)+g(\omega)]$$
$$+[q(\omega)+a(\omega)+a(\omega)q(\omega)+c(\omega)+c(\omega)q(\omega)+d(\omega)$$
$$+d(\omega)q(\omega)+f(\omega)p(\omega)+2f(\omega)p(\omega)q(\omega)]z_4(\omega)$$
$$+[f(\omega)q(\omega)+f(\omega)q^2(\omega)]z_4{}^2(\omega)$$
$$=A(\omega)+B(\omega)z_4(\omega)+C(\omega)z_4{}^2(\omega)$$
$$\omega\in[\omega_0,\infty) \tag{26}$$

where, $A(\omega),B(\omega),C(\omega)$ is defined by(9)~(11).

By replacing ω with υ in (26), then integrating both sides of the inequality with respect to υ from ω_0 to ω gives

$$z_4(\omega)\le z_4(\omega_0)+\int_{\omega_0}^{\omega}A(\upsilon)d\upsilon+\int_{\omega_0}^{\omega}B(\upsilon)z_4(\upsilon)d\upsilon+\int_{\omega_0}^{\omega}C(\upsilon)z_4{}^2(\upsilon)d\upsilon$$
$$\omega\in[\omega_0,\infty) \tag{27}$$

Since equation (27) satisfies the requirements of the lemma，an estimate of $z_4(\omega)$ in equation (27) could be obtained according to the lemma.

$$z_4(\omega)\le(\exp(-(\ln(z_4(\omega_0))+\int_{\omega_0}^{\omega}A(\upsilon)d\upsilon)-\int_{\omega_0}^{\omega}B(\upsilon)d\upsilon)-\int_{\omega_0}^{\omega}C(\upsilon)d\upsilon)^{-1}$$
$$\omega\in[\omega_0,\infty) \tag{28}$$

From (13),(16),(20),(24),we have

$$u(\omega_0)=z_1(\omega_0)=z_2(\omega_0)=z_3(\omega_0)=z_4(\omega_0) \tag{29}$$

Therefore

$$z_4(\omega)\le(\exp(-(\ln(u(\omega_0))+\int_{\omega_0}^{\omega}A(\upsilon)d\upsilon)-\int_{\omega_0}^{\omega}B(\upsilon)d\upsilon)-\int_{\omega_0}^{\omega}C(\upsilon)d\upsilon)^{-1}=Z(\omega)$$
$$\omega\in[\omega_0,\infty) \tag{30}$$

where, $Z(\omega)$ is defined by (8).

Substituting (30) into (25), gives

$$z_3'(\omega)\le p(\omega)[1+a(\omega)+c(\omega)+d(\omega)]$$
$$+[q(\omega)+a(\omega)+a(\omega)q(\omega)+c(\omega)+c(\omega)q(\omega)$$
$$+d(\omega)+d(\omega)q(\omega)]Z(\omega)$$
$$\omega\in[\omega_0,\infty) \tag{31}$$

Integrating both sides of (31) gives

$$z_3(\omega)\le z_3(\omega_0)+\int_{\omega_0}^{\omega}\{p(\upsilon)[1+a(\upsilon)+c(\upsilon)+d(\upsilon)]$$
$$+[q(\upsilon)+a(\upsilon)+a(\upsilon)q(\upsilon)+c(\upsilon)+c(\upsilon)q(\upsilon)$$
$$+d(\upsilon)+d(\upsilon)q(\upsilon)]Z(\upsilon)\}d\upsilon$$
$$\omega\in[\omega_0,\infty) \tag{32}$$

From (29) we know $u(\omega_0)=z_3(\omega_0)$, therefore we have

$$z_3(\omega)\le u(\omega_0)+\int_{\omega_0}^{\omega}\{p(\upsilon)[1+a(\upsilon)+c(\upsilon)+d(\upsilon)]$$
$$+[q(\upsilon)+a(\upsilon)+a(\upsilon)q(\upsilon)+c(\upsilon)+c(\upsilon)q(\upsilon)$$
$$+d(\upsilon)+d(\upsilon)q(\upsilon)]Z(\upsilon)\}d\upsilon$$
$$=u(\omega_0)+\int_{\omega_0}^{\omega}p(\upsilon)[1+a(\upsilon)+c(\upsilon)+d(\upsilon)]$$
$$+[q(\upsilon)(1+a(\upsilon)+c(\upsilon)+d(\upsilon))+a(\upsilon)+c(\upsilon)+d(\upsilon)]Z(\upsilon)\}d\upsilon$$

$$\omega\in[\omega_0,\infty) \tag{33}$$

From (13), (16), (20), (24), we have

$$u(\omega)\le z_1(\omega)\le z_2(\omega)\le z_3(\omega)\le z_4(\omega) \tag{34}$$

Therefore,

$$u(\omega)\le u(\omega_0)+\int_{\omega_0}^{\omega}p(\upsilon)[1+a(\upsilon)+c(\upsilon)+d(\upsilon)]$$
$$+[q(\upsilon)(1+a(\upsilon)+c(\upsilon)+d(\upsilon))+a(\upsilon)+c(\upsilon)+d(\upsilon)]Z(\upsilon)\}d\upsilon$$
$$\omega\in[\omega_0,\infty) \tag{35}$$

3. Application

In this section, we apply Theorem 1 to estimated the solutions of differential-integral equations. For example,

$$x'(\omega)=p(\omega)+q(\omega)\{x(\omega)+\int_{\omega_0}^{\omega}F(\xi,x(\xi),x'(\xi))d\xi\},\quad x(0)=c \tag{36}$$

where, $|c|$ is a normal number and $p(\omega),q(\omega)$ are defined by Theorem 1. $F\in C(R\times R\times R,R)$, and F satisfies the following conditions

$$|F(\omega,x_1,x_2)|\le a(\omega)\{|x_1|+|x_2|+\int_{\omega_0}^{\omega}|G(\upsilon,x_1,x_2)|d\upsilon\}$$

$$|G(\omega,x_1,x_2)|\le c(\omega)\{|x_1|+|x_2|+\int_{\omega_0}^{\omega}|H(\upsilon,x_1,x_2)|d\upsilon\}$$

$$|H(\omega,x_1,x_2)|\le d(\omega)\{|x_1|+|x_2|+\int_{\omega_0}^{\omega}|W(\upsilon,x_1,x_2)|d\upsilon\}$$

$$|W(\omega,x_1,x_2)|\le f(\omega)|x_2|[|x_1|+|x_2|+g(\omega)] \tag{37}$$

where $a(\omega),c(\omega),d(\omega),f(\omega),g(\omega)$ is defined by Theorem 1.

Suppose $|c|$, $a(\omega),c(\omega),d(\omega),f(\omega),g(\omega)$ satisfies the following conditions

$$\exp(-\ln(|c|+\int_{\omega_0}^{\omega}A(\xi)d\xi)-\int_{\omega_0}^{\omega}B(\xi)d\xi)-\int_{\omega_0}^{\omega}C(\xi)d\xi>0$$
$$\omega\in[\omega_0,\infty) \tag{38}$$

If $x(\omega)$ is the solution to equation (36), we can estimate the magnitude of $x(\omega)$.

$$|x(\omega)|\le|c|+\int_{\omega_0}^{\omega}\{p(\xi)[1+a(\xi)+c(\xi)+d(\xi)]+[q(\xi)+a(\xi)$$
$$+a(\xi)q(\xi)+c(\xi)+c(\xi)q(\xi)+d(\xi)+d(\xi)q(\xi)]\tilde{Z}(\xi)\}d\xi$$
$$\omega\in[\omega_0,\infty) \tag{39}$$

where

$$\tilde{Z}(\omega):=(\exp(-\ln(|c|+\int_{\omega_0}^{\omega}A(\xi)d\xi)-\int_{\omega_0}^{\omega}B(\xi)d\xi)-\int_{\omega_0}^{\omega}C(\xi)d\xi)^{-1} \tag{40}$$

$A(\omega),\ B(\omega),\ C(\omega)$ is defined by (9)~(11).

Proof. Using condition (37) and from equation (36), we have

$$|x'(\omega)|\le p(\omega)+q(\omega)\{|x(\omega)|+\int_{\omega_0}^{\omega}[a(\xi)(|x(\xi)|+|x'(\xi)|)$$
$$+a(\xi)\int_{\omega_0}^{\xi}[c(\sigma)(|x(\sigma)|+|x'(\sigma)|)+c(\sigma)\int_{\omega_0}^{\sigma}[d(\eta)((|s(\eta)|+|s'(\eta)|)$$
$$+d(\eta)\int_{\omega_0}^{\eta}[f(\varphi)|x'(\varphi)|(|x(\varphi)|+|x'(\varphi)|)+g(\varphi)]d\varphi]d\eta]d\sigma]d\xi\}$$
$$\omega\in[\omega_0,\infty) \tag{41}$$

Since equation (41) has the form of inequality (3) and satisfies the corresponding conditions in theorem 1, the estimator (39) of the solution module of equation (36) can be obtained by using theorem 1.

4. Conclusion

In this research, a class of four-fold nonlinear integral inequalities with unknown derivations has been discussed. Using various inequality analysis methods such as variable substitution, amplification, differentiation and integration, the unknown functions in four-fold integral inequalities were estimated. The obtained results can be applied to estimate differential equation solutions.

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