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Kinetic Analysis of Fractional Viscoelastic Films Based on Shifting Legendre Polynomials

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Abstracts: In this paper, the variable fractional derivative model was used to simulate the structure of viscoelastic polyethylene terephthalate (PET) films. the shift Legendre polynomial algorithm is used to directly solve the equation of motion in the time domain, and the nonlinear dynamic response of the fractional viscoelastic film is reliably estimated. Numerical examples demonstrate the reliability and accuracy of the proposed strategy, and the displacement of thin films with different aspect ratios under simple harmonic loading is numerically simulated.

Keywords: Fractional Order; Shift Legendre Polynomial Algorithm; Membrane; Numerical Simulation

1. Introduction

PET film is widely used in packaging, electronics, medical, construction and other fields, because of its high transparency, barrier, flexibility and environmental protection, it has become the preferred material for food packaging, electronic product protective film, medical equipment and building waterproofing membrane and other industries, showing its diversity and importance in modern industry. PET film is generally produced by roller-to-roller manufacturing, which is an advanced method of manufacturing electronic products with high production efficiency and excellent performance [1]. The film is conveyed through a drum during the production process and is used to achieve printing, coating and other processes. the pattern is continuously printed on the film on multiple rollers and registered. the process can lead to uneven surface distribution and nonlinear vibrations of the moving film, which may even affect the quality of the flexible product. Therefore, it is critical to accurately predict the nonlinear dynamics of mobile membranes to meet the stringent requirements of flexible manufacturing. the dynamics of moving materials has also been of interest to researchers [2].

Many scholars have studied the film: Jimei Wu et al. [3] explored the complex relationship between the elastic modulus and vibration characteristics of PET films in motion at different temperatures, revealed the influence of thermoviscoelastic coupling effect on the nonlinear vibration behavior, and studied the influence of film width, amplitude and frequency of external excitation, transmission speed, ambient temperature, system damping and viscoelastic coefficient on the nonlinear vibration of moving films. Zuocai Dai et al. [4] performed a dynamic analysis of the damped vibration of a lattice cylindrical shell of a lattice cylindrical shell such as a grid filled with viscoelastic foam perfectly.

Most of the current studies are based on integer-order and fractional-order models to establish governing equations, and then numerically simulate some physical phenomena, which cannot eliminate the error between the actual material properties and the experimental results [5]. However, for a composite flexible substrate such as PET, it is highly likely that large deformation will occur during the production process [6]. Therefore, a model that takes into account the viscoelasticity and large deformation of PET film is needed, so that we can more accurately simulate the physical phenomena in the production process, so as to have a beneficial impact on production. In order to describe the complex dynamic behavior of elastic materials in different working environments, fractional-order models and variable-order models that can better describe the memory properties of materials compared with traditional models have emerged. In this paper, we are committed to the dynamic analysis of thin films on rollers using a variational fractional order model.

The structure of this paper is as follows: Section 2 establishes the partial differential governing equations for variable fraction films. Section 3 converts the governing equations of thin films into matrix form based on the differential operation matrix of the shifted Legendre polynomial. Section 4 gives error analysis and mathematical examples of algorithms. Section 5 gives the conclusion of this paper.

2. Establishment of Nonlinear Differential Governing Equations for Viscoelastic Microbeams

In the manufacturing process of the film, it is assumed that the film is uniformly continuous, follows elasticity and isotropy, and is not affected by bending stiffness and shear forces due to its light weight and soft properties [7], in addition, the large deformation assumption should be satisfied. That is, compared with the original size, the deformation of the film under stress cannot be ignored. Figure 1 is a schematic representation of a viscoelastic rectangular film under external excitation. The membrane has a length of$L$and a width of$b$and a thickness of h , external incentives for $F.\cos(wt)$, and the film is delivered at velocity$v$. The governing equations of the thin film can be formulated as:

$$\rho\left(\frac{\partial^2\bar{w}}{\partial t^2}+2v\frac{\partial^2\bar{w}}{\partial x\partial t}+v^2\frac{\partial^2\bar{w}}{\partial x^2}\right)-N_x\frac{\partial^2\bar{w}}{\partial x^2}-N_y\frac{\partial^2\bar{w}}{\partial y^2}-\bar{T}_0(1+\beta\frac{y}{b})\frac{\partial^2\bar{w}}{\partial x^2}-\bar{F}\cos(\varpi t)=0 \quad (1)$$

where$\rho$is the density of the membrane. T_x is the variable axial tension, w indicates lateral displacement.

The following dimensionless quantities are introduced:

$$w=\frac{\bar{w}}{h},\bar{x}=\frac{x}{L},\bar{y}=\frac{y}{b},\tau=t\sqrt{\frac{Eh^3}{\rho L^4}},c=v\sqrt{\frac{\rho L^2}{Eh^3}},\zeta=\frac{1}{2(1-\mu^2)}$$

$$r=\frac{L}{b},F=\bar{F}\frac{L^4}{Eh^4},\eta=\bar{\eta}(\sqrt{\frac{Eh^3}{\rho L^4}})^{\alpha(t)},\omega=\varpi\sqrt{\frac{\rho L^4}{Eh^3}}$$

The above equation becomes:

$$-\zeta(1+\eta D^{\alpha(t)})\left[\left(\frac{\partial w}{\partial\bar{x}}\right)^2\frac{\partial^2 w}{\partial\bar{x}^2}+\mu r^2\left(\frac{\partial w}{\partial\bar{y}}\right)^2\frac{\partial^2 w}{\partial\bar{x}^2}+r^4\left(\frac{\partial w}{\partial\bar{y}}\right)^2\frac{\partial^2 w}{\partial\bar{y}^2}+\mu r^2\left(\frac{\partial w}{\partial\bar{x}}\right)^2\frac{\partial^2 w}{\partial\bar{y}^2}\right]$$
$$(\frac{\partial^2 w}{\partial\tau^2}+2c\frac{\partial^2 w}{\partial\bar{x}\partial\tau}+c^2\frac{\partial^2 w}{\partial\bar{x}^2})-T_0(1+\beta\bar{y})\frac{\partial^2 w}{\partial\bar{x}^2}-F\cos\omega\tau=0 \quad (2)$$

Boundary conditions can be expressed as:

$$\begin{cases}w(0,\bar{y},\tau)=w(1,\bar{y},\tau)=0\\ w(\bar{x},0,\tau)=w(\bar{x},1,\tau)=0\end{cases},\begin{cases}\frac{\partial^2 w(0,\bar{y},\tau)}{\partial\bar{x}^2}=\frac{\partial^2 w(L,\bar{y},\tau)}{\partial\bar{x}^2}=0\\ \frac{\partial^2 w(\bar{x},0,\tau)}{\partial\bar{y}^2}=\frac{\partial^2 w(\bar{x},b,\tau)}{\partial\bar{y}^2}=0\end{cases}$$

The initial condition can be expressed as [8]:

$$\begin{cases}w(\bar{x},\bar{y},\tau)|_{\tau=0}=0\\ \frac{\partial w(\bar{x},\bar{y},\tau)}{\partial\tau}|_{\tau=0}=0\end{cases}$$

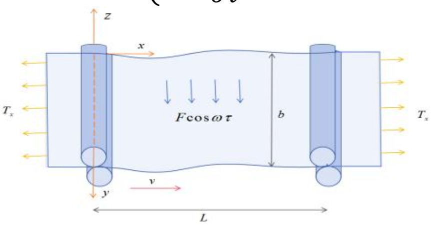


Figure 1. Schematic Diagram of a Thin Film at Simple Harmonic Forces

3. Functional Approximation of The Shift Legendre Polynomial

Definition 1: The Legendre polynomial, defined on interval $[0,1]$, is expressed as:

$$l_{n,i}(x)=\sum_{i=0}^{n}(-1)^{n+i}\frac{\Gamma(n+i+1)}{\Gamma(n-i+1)\left(\Gamma(i+1)\right)^2}x^i \quad (3)$$

Where $i=0,1,\ldots,n,x\in[0,1]$.

The column vector $\varphi(x)$ is constituted by the Legendre polynomial on the interval$[0,1]$, and it can be expressed as

$$\varphi(x)=[l_{n,0}(x),l_{n,1}(x),\ldots,l_{n,n}(x),]^T=AZ(x) \quad (4)$$

Where$Z(x)=[1,x,\cdots,x^n]^T$,

A is the Legendre polynomial coefficient matrix:

$$A_n=\begin{bmatrix}a_{0,0}&0&\cdots&0\\ a_{1,0}&a_{1,1}&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ a_{n,0}&a_{n,1}&\cdots&a_{n,n}\end{bmatrix},a_{ij}$$

$$= \begin{cases} (-1)^{i+j} \dfrac{\Gamma(n+i+1)}{\Gamma(n-i+1)\big(\Gamma(i+1)\big)^2}, i \ge j \\ 0, i < j. \end{cases}$$

Extending the Legendre polynomial from the interval $[0,1]$ to $[0,K]$ yields the shifted Legendre polynomial:

$$L_{n,i}(x) = \sum_{i=0}^{n} (-1)^{n+i} \frac{\Gamma(n+i+1)}{\Gamma(n-i+1)\big(\Gamma(i+1)\big)^2} \left(\frac{1}{K}\right)^i x^i \quad (5)$$

Where $i = 0,1,\dots,n, x \in [0,K]$.

At this point, $\varphi(x)$ can be represented by the displaced Legendre multinomial:

$$\varphi(x) = PZ(x) \quad (6)$$

$P$is the matrix of shifted Legendre polynomial coefficients:

$$P = \begin{bmatrix} p_{0,0} & 0 & \cdots & 0 \\ p_{1,0} & p_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,0} & p_{n,1} & \cdots & p_{n,n} \end{bmatrix}, p_{ij} = \begin{cases} (-1)^{i+j} \dfrac{\Gamma(n+i+1)}{\Gamma(n-i+1)\big(\Gamma(i+1)\big)^2} \left(\dfrac{1}{K}\right)^i, i \ge j \\ 0, i < j \end{cases}$$

The governing equation (2) for the thin film can be expressed as a matrix of variable-order differential operators:

$$\rho A \varphi^T(x) W K_t^2 \varphi(t) + (EI + \mu A l^2)\varphi^T(x) N_x^4 W \varphi(t) + [(EI + \mu A l^2)\eta_d]\{Z^T(t) U^T H^T W^T N_x \varphi(x) \varphi^T(x) N_x W[- \frac{3}{2} EA[\varphi^T(x)(PSP^{-1})^4 WHUH^{-1}\varphi(t)] - \frac{3}{2} EA\eta_d[\varphi^T(t) W^T N_x \varphi(x) \varphi^T(x) N_x W \varphi(t)][\varphi + c_0 \varphi^T(x) W K_t \varphi(t) = F(x,t) \quad (7)$$

Where $W = \begin{bmatrix} w_{0,0} & w_{0,1} & \cdots & w_{0,n} \\ w_{1,0} & w_{1,1} & \cdots & w_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n,0} & w_{n,1} & \cdots & w_{n,n} \end{bmatrix}, w_{ij} = \begin{cases} \frac{\Gamma(i)}{\Gamma(i+1)}, i = j, i \ne 1 \\ 0, otherwise \end{cases}, H = \begin{bmatrix} h_{0,0} & 0 & \cdots & 0 \\ h_{1,0} & h_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{n,0} & h_{n,1} & \cdots & h_{n,n} \end{bmatrix}$, N_x is the first-order differential operator matrix of the shifted Legendre polynomial. K_t is the first-order differential operator matrix of the shifted Legendre polynomial with respect to t.

4. Numerical Examples

In this section, it should be noted in advance that the parameters in numerical studies are arbitrary values and have no actual physical significance. Numerical calculations are illustrated in the following equation:

$$\frac{\partial^2 w}{\partial \tau^2} + \frac{\partial^2 w}{\partial \bar{x} \partial \tau} + \frac{\partial^2 w}{\partial \bar{x}^2} - \bar{y}\frac{\partial^2 w}{\partial \bar{x}^2} - \left(\frac{\partial w}{\partial \bar{x}}\right)^2 \frac{\partial^2 w}{\partial \bar{x}^2} + \left(\frac{\partial w}{\partial \bar{y}}\right)^2 \frac{\partial^2 w}{\partial \bar{x}^2} + \left(\frac{\partial w}{\partial \bar{y}}\right)^2 \frac{\partial^2 w}{\partial \bar{y}^2} + \left(\frac{\partial w}{\partial \bar{x}}\right)^2 \frac{\partial^2 w}{\partial \bar{y}^2} - D^{\alpha(t)}\left[\left(\frac{\partial w}{\partial \bar{x}}\right)^2 \frac{\partial^2 w}{\partial \bar{x}^2} + \left(\frac{\partial w}{\partial \bar{y}}\right)^2 \frac{\partial^2 w}{\partial \bar{x}^2} + \left(\frac{\partial w}{\partial \bar{y}}\right)^2 \frac{\partial^2 w}{\partial \bar{y}^2} + \left(\frac{\partial w}{\partial \bar{x}}\right)^2 \frac{\partial^2 w}{\partial \bar{y}^2}\right] = F \quad (8)$$

The boundary conditions are the same as above. Among them are: $\alpha(t) = 1 - 0.75t, x \in [0,1], y \in [0,1]$.

$$F(x,y,t) = 2x^2(1-x)^2y^2(1-y)^2 + 2t(2x - 6x^2 + 4x^3)y^2(1-y)^2 + (2x - 6x^2 + 4x^3)y^2(1-y)^2 2t + 0.35 * (2 - 12x + 12x^2)y^2(1-y)^2t^2$$
$$-y(2 - 12x + 12x^2)y^2(1-y)^2t^2 - [(2x - 6x^2 + 4x^3)y^2(1-y)^2t^2]^2(2 - 12x + 12x^2)y^2(1-y)^2t^2$$
$$-[x^2(l-x)^2(2y - 6y^2 + 4y^3)t^2]^2(2 - 12x + 12x^2)y^2(1-y)^2t^2 - [x^2(l-x)^2(2y - 6y^2 + 4y^3)t^2]^2x^2(1-x)^2(2 - 12y + 12y^2)t^2$$
$$-[(2x - 6x^2 + 4x^3)y^2(1-y)^2t^2]^2x^2(1-x)^2(2 - 12y + 12y^2)t^2$$
$$-[(2x - 6x^2 + 4x^3)y^2(1-y)^2]^2(2 - 12x + 12x^2)y^2(1-y)^2\frac{\Gamma(7)}{\Gamma(7-at)}t^{(6-at)} - [x^2(1-x)^2(2y - 6y^2 + 4y^3)]^2(2 - 12x + 12x^2)y^2(1-y)^2\frac{\Gamma(7)}{\Gamma(7-at)}t^{(6-at)}$$
$$-[x^2(1-x)^2(2y - 6y^2 + 4y^3)]^2x^2(1-x)^2(2 - 12y + 12y^2)\frac{\Gamma(7)}{\Gamma(7-at)}t^{(6-at)} - [(2x - 6x^2 + 4x^3)y^2(1-y)^2]^2x^2(1-x)^2(2 - 12y + 12y^2)\frac{\Gamma(7)}{\Gamma(7-at)}t^{(6-at)}$$

The exact solution

of the equation is:

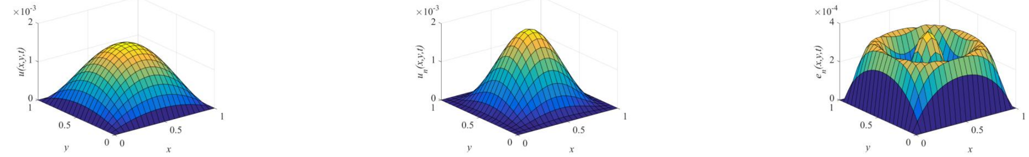
$$w(x,y,t) = x^2(1-x)^2y^2(1-y)^2t^2 \quad (9)$$

When $n = 2$, the shift Legendre polynomial algorithm is used to solve the equation (2). the numerical solution was performed using MATLAB programming and coordination method. $w_n(x,t)$ is the numerical solution, $w(x,t)$ is the exact solution, and $e(x,t)$ is the absolute error:

$$e(x,t) = |w(x,t) - w_n(x,t)| \quad (10)$$

Fig. 2 a-c are the analytical solution, numerical solution and absolute error of Eq. (8), respectively.

As can be clearly seen from Figure 2, the numerical solution is extremely accurate. the effectiveness and accuracy of the shift Legendre polynomial algorithm are verified. At the same time, it is also proved that the algorithm is suitable for the dynamic analysis of variable fractional viscoelastic films.



(a)Numerical solution (b)Real solution (c)absolute error

Figure 2. Solution of a Numerical Study

5. Conclusion

In this paper, the variable fractional nonlinear differential equations for viscoelastic films are established, and a numerical algorithm for solving the nonlinear variable fractional differential equations is proposed. This also lays the theoretical foundation for the development of more robust and efficient thin film structures.

References

[1] He K, Liu S, Wang K, et al. Thermal characteristics of plastic film tension in roll-to-roll gravure printed electronics [J]. Applied Sciences, 2018, 8(2):312.

[2] Banichuk N, Jeronen J, Neittaanmäki P, et al. Theoretical study on travelling web dynamics and instability under non-homogeneous tension [J]. International Journal of Mechanical Sciences, 2013, 66:132-140.

[3] Wu J, Shao M, Wang Y, et al. Stability analysis of moving printing web with sine half-wave variable density based on differential quadrature method [J]. Journal of Vibration and Acoustics, 2017, 139(6):061018.

[4] Dai, Zuocai, Yan Shi, and Yaser Kiani. "Free-damped vibration analysis of viscoelastic foam-filled FGM anisogrid lattice cylindrical shells. " Thin-Walled Structures 195(2024):111373.

[5] Xu, Jun, et al. "Vibration analysis of complex fractional viscoelastic beam structures by the wave method. " International Journal of Mechanical Sciences 167(2020):105204.

[6] Qing, Jiajuan, et al. "Nonlinear dynamics of fractional viscoelastic PET membranes with linearly varying density. " Communications in Nonlinear Science and Numerical Simulation 127(2023):107559.

[7] Shin CH, Kim WS, Chung JT. Free in-plane vibration of an axially moving membrane. J Sound Vib- J Sound Vib 2004; 272:137–54.

[8] Zheng ZL, Liu CY, Li D, Zhang T. Dynamic response of orthotropic membrane structure under impact load based on multiple scale perturbation method. Latin American Journal of Solids and Structures 2018; 14:1490–505.