Estimating Parameters of the Norris-Landzberg Model under Gamma Distribution

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Abstract: With the increase quality of products, the traditional life-testing method is too time-consuming. Therefore, the accelerated life-testing (ALT) is widely used. If the product operating conditions fluctuate continuously, the cyclic accelerated life-testing (CALT) is considered. Among CALTs, the Norris-Landzberg model is commonly used in modeling the growing crack resulting from the thermal cyclic stress. In the parameter estimate of this model, many researchers have employed linear regression models without a proper parametric assumption, which may impair its statistical integrity. Therefore, this paper propose a Gamma distribution assumption, which will ensure the unbiasedness and consistency of estimator. The corresponding inference is also shown.

Keywords: Norris-Landzberg Model; Linear Regression; Statistical Inference.

1.Introduction

In today's fiercely competitive global market, products are manufactured to meet stringent quality standards, resulting in extended product lifespans. Traditional life-testing methods, which rely on observing failures under standard operating conditions, can be impractical due to the lengthy period required to collect meaningful data. To address this challenge, accelerated life-testing (ALT) techniques have become a popular approach to expedite the data collection process. Comprehensive discussions on ALT can be found in the literature [1-6].

While the majority of research has concentrated on constant stress ALT, the cyclic accelerated life-testing (CALT) method has been introduced for scenarios where product operating conditions fluctuate continuously. This approach has been successfully applied across various industries, including the reliability assessment of engines [7] and electric drive systems [8]. CALT incorporates several factors such as temperature cycling, vibration, and humidity fluctuations. Notably, the Coffin-Manson model [9], which focuses on the temperature range, is commonly used for modeling temperature cycling. However, this model does not account for the maximum temperature and cycling frequency, which are also critical parameters. To provide a more comprehensive framework for CALT analysis, the Norris-Landzberg model [10] has been proposed. Extensive researches are based on the Norris-Landzberg model, see [11-13].

In the parameter estimation process of the Norris-Landzberg model, many researchers have employed linear regression models without a proper parametric assumption, leading to estimators that may lack statistical robustness. This could result in biased and inconsistent estimators. To ensure the statistical integrity of the least squares method, we introduce a Gamma distribution assumption in this paper. It is observed that the shape parameter of the Gamma distribution does not affect the inference of the Norris-Landzberg model, hence the shape parameter is assumed to be known.

The structure of this paper is as follows: Section 2 outlines the methodology, including the Norris-Landzberg model and the least squares method with Gamma distribution family. Section 3 details the simulation and analysis procedures, covering simulation settings, the accuracy of estimates, the asymptotic normality of estimates, and the evaluation of normal confidence intervals. Finally, Section 4 presents the concluding remarks.

2.Methodology

2.1 Norris-Landzberg model

The Norris-Landzberg model, proposed by [10], is widely used in modeling the growing crack resulting from the thermal cyclic stress. It assumes that the number of cycle to failure, denoted by N, is influence by the temperature range ΔT, the highest temperature in the cycle T_{max}, and the cycling frequency f. The relationship is shown in the following formula

$$\boldsymbol{N = Af^{-a}(\Delta T)^{-b}G(T_{max})} \qquad (1)$$

where $A > 0$ represents the constant depends on the material,

a represents the exponent of cycling frequency, typically near $-\frac{1}{3}$,

b represents the exponent of the temperature range, which depends on the test environment,

$G(T_{max}) = exp\{\frac{E_a}{K^* T_{max}}\}$ represents Arrhenius term with $T = T_{max}$, where $K^* = 8.623 \times 10^5 eV/K$ is the Boltzmann's constant and E_a is the activation energy determined by experiment.

We also use the reduce model like Coffin-Manson-Arrhenius model and Coffin-Manson model in practice. The Coffin-Manson-Arrhenius model

$$N = A(\Delta T)^{-b}G(T_{max}) \qquad (2)$$

exclude the cycling frequency term since $a = 0$. It can be further reduced to the Coffin-Manson model

$$N = A(\Delta T)^{-b} \qquad (3)$$

by taking $E_a = 0$. The subsequent discussion will focus on the realistic case with $a \leq 0, b \geq 0$ and $E_a \geq 0$.

2.2 Least Square Method with Gamma Distribution Family

It is assumed that the real failure time T is visible for all test units. Then the formulation of the Norris-Landzberg model is changed by

$$\boldsymbol{T_{ij} = \frac{N_{ij}}{f_i} = Af_i^{-(a+1)}(\Delta T_i)^{-b}G(T_{i,max}), i = 1,2,\ldots,n, j = 1,2,\ldots,N_i} \qquad (4)$$

where $\{T_{ij} | i = 1,2,\ldots,n, j = 1,2,\ldots,N_i\}$ represent the ith test environment's failure time. The ith stress factor, denoted by $S_i = [f_i, \Delta T_i, T_{i,max}]$, represents the strength of stress in the ith test environment. The stresses in the test environments influence the lifetimes of test units, which motive us to use a Gamma distribution family $Gamma(k, \beta_i)$ to formulate the lifetime distributions. Here k is assume to be known and β_i is assumed to be relevant to the ith test environment's stresses. Since the shape parameter is not influence by the environment's stresses, the log-linear model

$$\boldsymbol{\ln(t_{ij}) = \ln(A) - (a+1)\ln(f_i) - b\ln(\Delta T_i) + E_a \frac{1}{K^* T_{i,max}} + \epsilon_{ij}, i = 1,2,\ldots,n, j = 1,2,\ldots,N_i} \quad \textbf{(5)}$$

satisfies the conditions of Gauss-Markov Theorem. Hence, taking $\boldsymbol{\theta} = [\ln(A), (a+1), b, E_a]$ as the parameter vector, we can derive the loss function

$$\boldsymbol{Q(\theta) = \sum_{i=1}^{n}\sum_{j=1}^{N_i}(\ln(t_{ij}) - \widehat{\ln(t_{ij})})^2} \quad \textbf{(6)}$$

where $\widehat{\ln(t_{ij})} = \ln(A) - (a+1)\ln(f_i) - b\ln(\Delta T_i) + E_a \frac{1}{K^* T_{i,max}}$ is the estimated value of $\ln(t_{ij})$. By the least squares method, we can minimize $Q(\boldsymbol{\theta})$ the by taking $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$.

3. Simulation and Analysis

1. This section discusses the simulation and analysis procedure, which is divided into 4 subsections:
2. The simulation settings.
3. The accuracy of estimates.
4. The asymptotic normality of estimates
5. The evaluation of normal confidence intervals

3.1 The Simulation Settings

Table 1. Settings Used in the Simulation Study

Setting	Norris–Landzberg model							
	i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8
f	72	48	72	48	72	48	72	48
ΔT	120K	100K	100K	120K	120K	100K	100K	120K
T_{max}	393K	373K	373K	393K	373K	393K	393K	373K
Parameter setting								
$E_a = 0.17Ev,\ A = 10^4 = exp(9.2103),\ a = -\frac{1}{3},\ b = 1.9$								
Setting	Coffin–Manson–Arrhenius model							
	i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8

f	72	48	72	48	72	48	72	48
ΔT	120K	100K	100K	120K	120K	100K	100K	120K
T_{max}	393K	373K	373K	393K	373K	393K	393K	373K
Parameter setting								
$E_a = 0.17Ev,\ A = 10^4 = exp(9.2103),\ a = 0,\ b = 1.9$								
Setting	Coffin–Manson model							
	i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8
f	72	48	72	48	72	48	72	48
ΔT	120K	100K	100K	120K	120K	100K	100K	120K
T_{max}	393K	373K	373K	393K	373K	393K	393K	373K
Parameter setting								
$E_a = 0Ev,\ A = 10^4 = exp(9.2103),\ a = 0,\ b = 1.9$								
Setting	Normal Condition							
$f = 1, \Delta T = 30K, T_{max} = 303K$								

We consider the Norris-Landzberg model along with the degraded cases of the Coffin-Manson-Arrhenius model and the Coffin-Manson model. The detailed settings are shown in the Table 1. The Gamma distribution $Gamma(6, \beta_i)$ is utilized in the simulation, while k=6 is the fixed shape parameter and β_i represents the scale parameter in ith test group. For the Gamma distribution, the scale parameters β_i for $i = 1,2,\ldots,n$ are expressed as

$$\beta_i = A\exp(-(a+1)\ln(f_i) - b\ln(\Delta T_i) + E_a \frac{1}{K^* T_{i,max}} - \psi(k)) \quad (7)$$

where $\psi(k)$ represents the digamma function evaluated at k.

3.2 The Accuracy of Estimates

Table 2. Bias and MSE of the estimators

$Gamma(6, \beta_i)$		Norris-Landzberg model			
		A	a	b	E_a
$N_i = 20$	Bias	-0.0303	-0.0034	-0.0011	0.0003
	MSE	5.1401	0.0274	0.1345	0.0018
$N_i = 40$	Bias	0.0227	-0.0038	0.0009	0.0001
	MSE	2.6920	0.0132	0.0725	0.0009
$N_i = 80$	Bias	-0.0049	-0.0028	0.0027	0.0002
	MSE	1.2432	0.0066	0.0342	0.0005
$Gamma(6, \beta_i)$		Coffin-Manson-Arrhrnius model			
		A	a	b	E_a
$N_i = 20$	Bias	0.0753	/	0.0147	-0.0002
	MSE	4.7756	/	0.1398	0.0018
$N_i = 40$	Bias	-0.0386	/	-0.0110	0.0007
	MSE	2.3592	/	0.0673	0.0009
$N_i = 80$	Bias	0.0058	/	-0.0015	-0.0004
	MSE	1.1960	/	0.0346	0.0004
$Gamma(6, \beta_i)$		Coffin-Manson model			
		A	a	b	E_a
$N_i = 20$	Bias	-0.0221	/	-0.0048	/
	MSE	3.1532	/	0.1429	/
$N_i = 40$	Bias	-0.0607	/	-0.0032	/
	MSE	1.4714	/	0.0667	/
$N_i = 80$	Bias	-0.0160	/	-0.0034	/
	MSE	0.7774	/	0.0353	/

In this section, the mean square error (MSE) is used to evaluate the performance of $\hat{\theta}$, defined as:

$$MSE(\hat{\theta}) = Bias^2(\hat{\theta}) + Var(\hat{\theta}). \quad \textbf{(8)}$$

The simulation was performed 10,000 times with $N_i = 20,40,80$ respectively and the findings were meticulously analyzed and detailed in Table 2. As the quantity of units increases, both the bias and the MSE for each model show a decline, signifying that the estimator is both unbiased and consistent.

3.3 The asymptotic normality of estimates

If the condition of Gauss-Markov Theorem is satisfied, the estimators derived by the least squares method have the asymptotic normality. When the lifetimes follow the Gamma family, the G-M Theorem is satisfied, leading to the asymptotic normality of estimators. It can also be verified by the QQ-plots in the Figure 1. It shows the QQ-plots of each estimator when $N_i = 20,40,80$. It can be seen that even in the case that $N_i = 20$, the normality of estimators still maintain, which is a basis of the normal confidence interval of parameters.

3.4 The evaluation of the normal confidence intervals

Table 3. Coverage Probabilities of the Normal CIs

$Gamma(6, \beta_i)$		Norris-Landzberg model			
		A	a	b	E_a
$N_i = 20$	90%	0.8945	0.9001	0.9023	0.9045
	95%	0.9427	0.9456	0.9502	0.9503
$N_i = 40$	90%	0.8926	0.8977	0.8976	0.8952
	95%	0.9562	0.9512	0.9504	0.9521

$N_i = 80$	90%	0.8997	0.8988	0.9006	0.9005
	95%	0.9501	0.9502	0.9488	0.9498
Gamma$(6, \beta_i)$		Coffin-Manson-Arrhrnius model			
		A	a	b	E_a
$N_i = 20$	90%	0.8902	/	0.8859	0.8965
	95%	0.9436	/	0.9456	0.9478
	90%	0.9002	/	0.9005	0.8907
$N_i = 40$	95%	0.9502	/	0.9468	0.9487
$N_i = 80$	90%	0.9002	/	0.9032	0.8975
	95%	0.9523	/	0.9506	0.9524
Gamma$(6, \beta_i)$		Coffin-Manson model			
		A	a	b	E_a
$N_i = 20$	90%	0.8897	/	0.8995	/
	95%	0.9559	/	0.9486	/
$N_i = 40$	90%	0.8992	/	0.9503	/
	95%	0.9500	/	0.9507	/
$N_i = 80$	90%	0.9048	/	0.9012	/
	95%	0.9527	/	0.9534	/

When the normality of estimators is satisfied, we can use normal confidence interval to make interval estimate by formula

$$[\underline{\theta_k}, \overline{\theta_k}] = [\widehat{\theta_k} \pm z_{1-\frac{\alpha}{2}}\sqrt{[(A^TA)^{-1}]_{k,k}}] \quad \textbf{(9)}$$

where A represent coefficient matrix. The simulation was performed 10,000 times with $N_i = 20,40,80$ respectively and we calculate the coverage probability of the 90% and 95% confident interval. The detailed results are shown in the Table 3. It can be seen that the coverage probabilities in every groups are consistent with the nominal levels, that means, the normal interval is effective in the confidence interval construction.

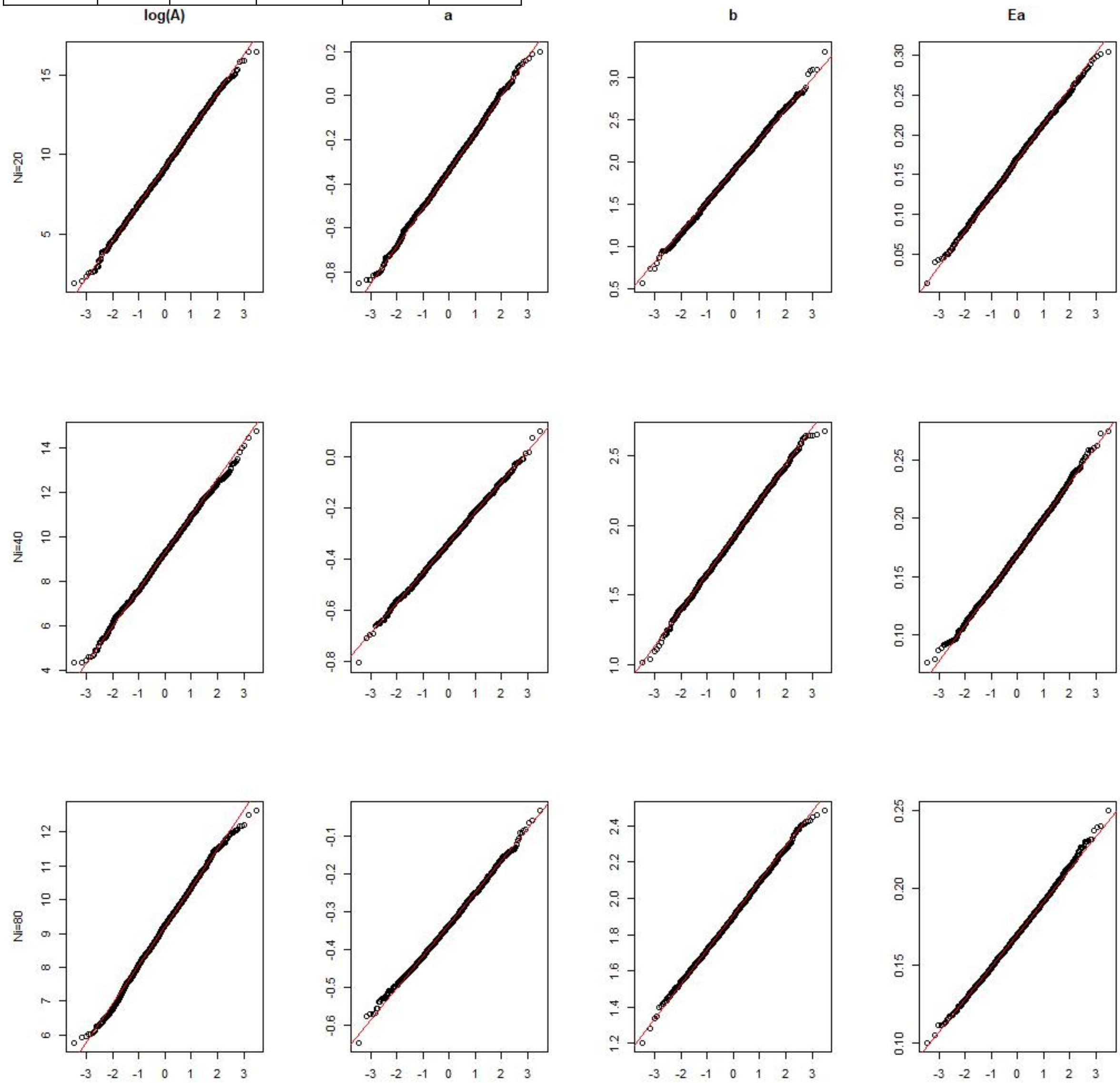

Figure 1. QQ-Plot of Parameters

4. Conclusions

This paper propose a Gamma distribution family assumption, and under this assumption the least squares method gives unbiased and consistent estimators of parameters, which is a statistically satisfactory results. If such a assumption is satisfied, the least squares method will be effective. In addition, this paper also explain the asymptotic normality of estimators, which ensures the effectiveness of the normal confidence interval. In reality, there are many other hypotheses with the same nature, which will become our future topic.

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