# The Equivalence Classes of Counting

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**Abstract： In this paper, we discuss the number of equivalence classes when a permutation group acts on a finite set which consist of mappings. First, we utilize some general permutation subgroups to act on a finite set consists of injective mappings. Next, we extend the case of injective mappings to all mappings from a finite set to a finite set. Moreover, we show the case when some general permutation subgroups act on a finite set consists of some special mappings from a finite set to a finite set. Finally, we give some applications of this topic.**

**Keywords: Equivalence Class; Permutation Group; Group Action; Mappings From a Finite Set To a Finite Set**

## 1. Introduction

In this paper, we discuss the number of equivalence classes when a permutation group acts on a finite set which consist of mappings. In many cases, we are not interested in the number of objects, but rather the number of equivalence classes of objects with respect to an appropriate equivalence relation. Moreover, these equivalence relations are often induced by certain permutation groups in a natural way. [4, Chapter 37] gives four examples for the number of equivalence classes of mappings. However, they only discuss the cases of equivalence relations induced by cyclic and dihedral permutation subgroups. Generally, there are many other permutation subgroups.

As we shown above, in this paper, we introduce some general permutation subgroups to act on a finite set consists of mappings. Then we obtain some general equivalence relations on this finite set. It's natural and logical to discuss the number of equivalence classes of these general equivalence relations. In Section 3.1, we utilize some general permutation subgroups to act on a finite set consists of injective mappings from a finite set to another finite set. Next, we utilize some general permutation subgroups to act on a finite set consists of all mappings from a finite set to another finite set in Section 3.2. Moreover, we show the case when some general permutation subgroups act on a finite set consists of some special mappings in Section 3.3. Finally, we give some applications of this topic in Section 4.

## 2. Preliminaries and Background

In this section, we review some preliminaries and notations of group theory in [1, 3, 4].

### 2.1 Preliminaries

First, we review some basic facts of set theory. For a nonempty set A, the number of elements in A will be denoted by |A| . A equivalence relation on A is a relation that holds between certain pairs of A. We may write it as a ~ b and speak of it as equivalence of a and b. An equivalence relation is required to be:

• reflexive: For all a ∈ A, a ~ a.
• symmetric: If a ~ b, then b ~ a.
• transitive: If a ~ b and b ~ c, then a ~ c.

Moreover, for any a ∈ A, the equivalence class of a ∈ A is defined to be {x ∈ A | x ~ a}. Also, a partition of A is a collection {Ai ≤ A | i ∈ I}, where I is an indexing set and

• A = ui∈IAi ,
• Ai ∩ Aj = 0 , for all i,j ∈ I with i ≠ j.

If ~ is an equivalence relation on A, then the set of all equivalence classes form a partition of A. Conversely, for any partition of A, the corresponding equivalence relation is defined by the rule that a ~ b if a and b lie in the same subset of the partition.

Next, we introduce some concepts of groups. A set G with a binary operation * is called a **group** if the following conditions are satisfied:

1. The operation * is closed, i.e. a * b ∈ G for all a,b ∈ G.
2. The operation * is associative, i.e. (a * b) * c = a * (b * c) for all a,b,c ∈ G.
3. There exists an identity element e ∈ G such that a * e = e * a = a for all a ∈ G.
4. For every element a ∈ G, there exists an inverse a −1 ∈ G such that a * a −1 = a −1 * a =

e.

Moreover, a group is called **abelian** if the operation * is commutative, i.e. a * b = b * a for all a,b ∈ G. Also, the **order** of G, denoted |G|, is defined as the number of elements in G. If |G| < ∞, then G is a finite group. Furthermore, a nonempty subset H of G with binary operation * is a **subgroup** of G if H is closed under products and inverses, that is,

- for any a,b ∈ H , a * b ∈ H ,
- if a ∈ H, then $a^{-1}$ ∈ H.

If $\{H_i : i \in I\}$ is a nonempty family subgroups, then $\cap_{i\in I}H_i$ is a subgroup of G. For any nonempty subset K of G, let $\{H_i : i \in I\}$ be the family of all subgroups of G which contain K. Then $\cap_{i\in I}H_i$ is called the subgroup of G generated by the set K and denoted (K〉 .

Finally, we introduce some facts of group actions. For a nonempty finite set A and a finite group G, an action of the group G on the set A is a function ϕ : G × A → A satisfying the following conditions:

- ϕ(e,a) = a for all a ∈ A, where e is the identity of G.
- $\phi(g_1, \phi(g_2, a)) = \phi(g_1g_2, a)$ for all a ∈ A and $g_1$ , $g_2$ ∈ G. Moreover, let ~ be a binary operation on set A defined by

a ~ b if and only if a = ϕ(g, b) for some g ∈ G

Then the relation ~ is an equivalence relation on A. Therefore, the equivalence classes of ~ forms a partition of set A. As for the number of equivalence classes of ~, we have the following lemma

**Lemma 2.1** (Burnside's Lemma)**.** Let G be a finite group acting on a finite set A, then the number of equivalence classes of ~ is given by

$$\frac{1}{|G|}\sum_{g\in G}|Fix(g)|,$$

where Fix(g) = {a ∈ A | ϕ(g, a) = a} is called the set of **fixed points** of g ∈ G.

### 2.2 Formulation

In this subsection, we formulate the our main problem of this paper. For simplicity, we only discuss mappings
from a finite ordered set X = {1, 2,..., r} to afinite set $Y = \{y_1 , y_2,..., y_n \}$ . We denote the set of all mappings from X to Y as $F_{r,n}$ , and $F_{r,n}$ has $n^r$ elements. Next, we discuss some special types of mappings in $F_{r,n}$. A mapping a : X → Y is injective if and only if $x_1 \neq x_2$ implies $a(x_1) \neq a(x_2)$ for any $x_1$ , $x_2$ ∈ X . Also, a mapping a : X → Y is surjective if and only if for each y ∈ Y , there is a x ∈ X such that a(x) = y. A mapping a : X → Y is a permutation if and only if a is surjective and injective. If a is a permutation, then r = n. However, the converse is not true.

Let $S_r$ be the set of all permutations from X onto X itself with a binary operation of composition ∗ . Then $S_r$ is a group which is called the **symmetric group** on the set X . For a π ∈ $S_r$ and pairwise different elements $x_1$ , $x_2$ , ..., $x_s$ ∈ X, if

$\pi(x_i) = x_{i+1}, i = 1, 2, ..., s - 1, \pi(x_s) = x_1 , \pi(x) = x, x \in X\backslash\{x_1 , x_2,..., x_s\}$ ,

then we call π the s-cyclic permutation, which is denoted by $(x_1\ x_2...x_s)$, or $\pi_s$ for short. Each permutation g ∈ $S_r$ can be represented as the composition of some pairwise disjoint cyclic permutations in $S_r$ , that is, there exist some pairwise disjoint cyclic permutations $\pi_{10}, ...,\pi_{1d_1}, \pi_{20}, ...,\pi_{2d_2}, ...,\pi_{r0}, ...,\pi_{rd_r} \in S_r$ such that

$g = \pi_{10} * ...\pi_{1d_1} * \pi_{20} * ... * \pi_{2d_2} * ... * \pi_{r0} * ... * \pi_{rd_r}$ ,

where $1d_1 + 2d_2 + ... + rd_r = r$ , $\pi_{ij_1} \neq \pi_{ij_2}$ , $\pi_{i0} = (i)$, i = 1, 2, ...,r , $j_1 , j_2 = 1, 2,..., d_i$ , and $j_1 \neq j_2$ . We call g has the form of $1^{d_1} 2^{d_2} ...r^{d_r}$ . Moreover, we call g is an even permutation if only and if $r - (d_1 + d_2 + ... + d_r)$ is an even number. Otherwise, we call g odd permutation.

For a nonempty subset $M_{r,n}$ of $F_{r,n}$ and a nonempty subgroup H of $S_r$ , we introduce an action ϕ : H × $M_{r,n}$ → $M_{r,n}$ , and for any g ∈ H and a ∈ $M_{r,n}$ ,

(ϕ(g, a))(x) = a(g(x)), ∀x ∈ X.

Thus, we obtain a equivalence relation ~ and a partition of $M_{r,n}$. According to Lemma 2.1, when the subgroup H acts on $M_{r,n}$ , the number of equivalence classes are

$$\frac{1}{|H|}\sum_{g\in H}|Fix(g)| ,$$

where Fix(g) = {a ∈ $M_{r,n}$ | a(g(x)) = a(x), ∀x ∈ X} . In the next section, we discuss the number of equivalence classes of ~ with some special subsets $M_{r,n}$ and subgroups H.

## 3. Main Result

In this section, we discuss the number of equivalence classes of ~ with some subsets of $F_{r,n}$ and subgroups of $S_r$ . First, we introduce how to obtain subgroups in $S_r$ . For a subset K

of Sr , we can build the subgroup ⟨ K⟩ . Specially, for any element g ∈ Sr , ⟨ g⟩ is a general cyclic subgroup in Sr .
Next, we list some subgroups of symmetric group Sr (**permutation groups**).
1. Ir = ⟨ (1)⟩ is the **identity subgroup** of Sr and has only one element which is the identity.
2. Cr = ⟨ (12 ... r)⟩ is a **cyclic group** of Sr and has r elements. Each element in Cr has the form of $d^{\frac{r}{d}}$ . If d | r, then there exists φ(d) elements of the form $d^{\frac{r}{d}}$ in Cr , where φ is the Euler function. If d ∤ r , then Cr does not have an element of the form $d^{\frac{r}{d}}$ .
3. D2r = ⟨ (12 ... r), (r(r − 1) ... 1)⟩ is a **dihedral group** of Sr (r ≥ 3), and it has 2r elements. Cr is a subgroup of D2r, which has r elements. As for the remaining r elements, we have the following two cases:
• Ifr is an odd integer, then there exists r elements of the form $12^{\frac{r-1}{2}}$ in Cr .
• If r is an even integer, then there exists $\frac{r}{2}$ elements of the form $122^{\frac{r-2}{2}}$, and $\frac{r}{2}$ elements of the form $2^{\frac{r}{2}}$ in Cr .
4. Sr itself is a trivial subgroup and has r! elements. For any integer solution of 1d1 + 2d2 + ... + rdr = r ,
there exists
r!
d1 !d2 !...dr!1d1 2d2 ...rdr
elements of the form 1d1 2d2 . . . rdr .
5. Ar = ((123), (124), ..., (12r) ⟩ is the **alternating group** consists of all even permutations in Sr (r ≥ 3). It has $\frac{r!}{2}$ elements. For any integer solution of $1d_1 +$
an even number, there exists
r!
d1 !d2 !...dr!1d1 2d2 ...rdr
elements of the form 1d1 2d2 . . . rdr .
Next, we define Sk be the subset Sr , and for any g ∈ Sk , g1 is a permutation from {1, 2,..., k} onto {1, 2,..., k} and
g1 (x) = x, x ∈ {k + 1, k + 2,..., r}. (1)
On the other hand, we define Sr−k be the subset of Sr , and for any g2 ∈ Sk , g2 is a permutation from
{k + 1, k + 2,..., r} onto {k + 1, k + 2,..., r} and
g2 (x) = x, x ∈ {1, 2,..., k}. (2)
By definition, it is clear that Sk is isomorphic to Sk and Sr−k is isomorphic to Sr−k, respectively. Also,
we define Ck be the subset of Sr , and for any g1 ∈ Ck , g1 is a k-cyclic permutation from {1, 2,..., k} onto
{1, 2,..., k} and
g1 (x) = x, x ∈ {k + 1, k + 2,..., r}. (3)
On the other hand, we define Cr−k be the subset of Sr , and for any g2 ∈ Cr−k , g2 is a permutation from {k + 1, k + 2,..., r} onto {k + 1, k + 2,..., r} and
g2 (x) = x, x ∈ {1, 2,..., k}. (4)
By definition, it is clear that Ck is isomorphic to Ck and Cr−k is isomorphic to Cr−k, respectively. Similarly, we can define D2k , D2r−2k , Ak and Ar−k, where k ≥ 3 and r — k ≥ 3.
For any subgroup H1 in Sk and subgroup H2 in Sr−k, since H1 * H2 = H2 * H1 and H1 ∩ H2 = {(1)},
H1 *H2 is the inner direct product of H1 and H2 in Sr . Thus, it follows that H1 *H2 has |H1 ||H2 | elements. By the subgroups listed above, we can induce more subgroups of Sr by inner direct product operation. For example,
6. Ck * Cr−k = ((12...k), ((k + 1)(k + 2)...r)⟩ has k(r — k) elements.
7. D2k * D2r−2k = ((12...k), (k(k — 1)...1), ((k + 1)(k + 2)...r), (r(r — 1)...(k + 1))⟩ has 4k(r — k) elements. We consider the following four cases:
8. Ak * Ar−k = ((123), (124), ..., (12k), ((k + 1)(k + 2)(k + 3)), ..., ((k + 1)(k + 2)r) ⟩ has $\frac{k!(r-k)!}{4}$ elements.
9. Ck * D2r−2k = ((12...k), ((k + 1)(k + 2)...r), (r(r — 1)...(k + 1))⟩ has 2k(r — k) elements.
10. Sk * Sr−k has k!(r — k)! elements.
When an inner direct product group H1 * H2 acts on Mr,n , if a is equivalent to b in Mr,n , then there exists g ∈ H1 * H2 such that g = g1 * g2 , where g1 ∈ H1 , g2 ∈ H2 and
a(g(x)) = b(x), Yx ∈ X.
Since g1 ∈ H1 and g2 ∈ H2 , by (1) and (2), it follows that
a(g1 (x)) = b(x), ∀x ∈ {1, 2,.., k},
a(g2 (x)) = b(x), ∀x ∈ {k + 1, k + 2,..., r} .
Conversely, it is easy to check that a is equivalent to bin Mr,n. Therefore, the number of equivalence classes of Mr,n are

$\left(\frac{1}{|H_1|}\sum_{g1\in H1}|Fix(g1)|\right)$, $\left(\frac{1}{|H_2|}\sum_{g2\in H2}|Fix(g2)|\right)$,.

### 3.1 The Set of all Injections from X to Y

In this subsection, we discuss the set of all injections from X to Y, which is denoted by $X_{r,n}$. If $r > n$, then $X_{r,n} = \varnothing$. Otherwise, $|X_{r,n}| = \frac{n!}{(n-r)!}$. Thus, we only the case when $r \le n$. When the subgroup H acts on $X_{r,n}$, for any $g \in H$, if $a \in X_{r,n}$ is a fixed point of g, then $a(g(x)) = a(x)$ for any $x \in X$. Since a is an injection from X to Y, it follows that $g(x) = x$ for any $x \in X$, which ensures that g is an identity mapping. Therefore, when the subgroup H acts on $X_{r,n}$, only the identity in H has fixed points. Moreover, if g is the identity mapping, then for any $a \in X_{r,n}$, a is a fixed point of g. Thus, the total number of fixed points of the identity mapping are $\frac{n!}{(n-r)!}$. By Lemma 2.1, the number of equivalence classes are $\frac{1}{|H|}\sum_{g\in H}|Fix(g)| = \frac{1}{|H|}\frac{n!}{(n-r)!}$.

Next, we use some special subgroups H acts on $X_{r,n}$.

1. When $I_r$ acts on $X_{r,n}$ $(n \ge r)$, since $|I_r| = 1$, there are $\frac{n!}{(n-r)!}$ equivalence classes.
2. When $C_r$ acts on $X_{r,n}$ $(n \ge r)$, since $|C_r| = r$, there are $\frac{n!}{r(n-r)!}$ equivalence classes.
3. When $D_{2r}$ acts on $X_{r,n}$ $(n \ge r \ge 3)$, since $|D_{2r}| = 2r$, there are $\frac{n!}{2r(n-r)!}$ equivalence classes.
4. When $S_r$ $(n \ge r)$ acts on $X_{r,n}$, since $|S_r| = r!$, there are $\frac{n!}{r!(n-r)!}$ equivalence classes.
5. When $A_r$ acts on $X_{r,n}$ $(n \ge r \ge 3)$, since $|A_r| = \frac{r!}{2}$, there are $\frac{2n!}{r!(n-r)!}$ equivalence classes.
6. When $C_k * C'_{r-k}$ acts on $X_{r,n}$ $(n \ge r > k)$, since $|C_k * C'_{r-k}| = k(r-k)$, there are $\frac{n!}{k(r-k)(n-r)!}$ equivalence classes.
7. When $D_{2k} * D'_{2r-2k}$ acts on $X_{r,n}$ $(n \ge r > k \ge 3)$, since $|D_{2k} * D'_{2r-2k}| = 4k(r-k)$, there are $\frac{n!}{4k(r-k)(n-r)!}$ equivalence classes.
8. When $A_k * A'_{r-k}$ acts on $X_{r,n}$ $(n \ge r > k \ge 3, r-k \ge 3)$, since $|A_k * A'_{r-k}| = \frac{k!(r-k)!}{4}$, there are $\frac{4n!}{k!(r-k)!(n-r)!}$ equivalence classes.
9. When $C_k * D'_{2r-2k}$ acts on $X_{r,n}$ $(n \ge r > k \ge 3, r-k \ge 3)$, since $|C_k * D'_{2r-2k}| = 2k(r-k)$, there are $\frac{n!}{2k(r-k)(n-r)!}$ equivalence classes.
10. When $S_k * S'_{r-k}$ acts on $X_{r,n}$ $(n \ge r > k)$, since $|S_k * S'_{r-k}| = k!(r-k)!$, there are $\frac{n!}{k!(r-k)!(n-r)!}$ equivalence classes.

**Example 3.1.** If $r = 7$, $k = 3$ and $n = 8$, then we have that

1. When $I_7$ acts on $F_{7,8}$, there are $\frac{8!}{(8-7)!} = 40320$ equivalence classes.
2. When $C_7$ acts on $F_{7,8}$, there are $\frac{8!}{7\times(8-7)!} = 5760$ equivalence classes.
3. When $D_{14}$ acts on $F_{7,8}$, there are $\frac{8!}{2\times7\times(8-7)!} = 2880$ equivalence classes.
4. When $S_7$ acts on $F_{7,8}$, there are $\frac{8!}{7!\times(8-7)!} = 8$ equivalence classes.
5. When $A_7$ acts on $F_{7,8}$, there are $\frac{2\times8!}{7!\times(8-7)!} = 16$ equivalence classes.
6. When $C'_3 * C'_4$ acts on $F_{7,8}$, there are $\frac{8!}{3\times(7-3)\times(8-7)!} = 3360$ equivalence classes.
7. When $D'_6 * D'_8$ acts on $F_{7,8}$, there are $\frac{8!}{4\times3\times(7-3)\times(8-7)!} = 840$ equivalence classes.
8. When $A'_3 * A'_4$ acts on $F_{7,8}$, there are $\frac{4\times8!}{3!\times(7-3)!\times(8-7)!} = 1120$ equivalence classes.
9. When $C'_3 * D'_8$ acts on $F_{7,8}$, there are $\frac{8!}{2\times3\times(7-3)\times(8-7)!} = 1680$ equivalence classes.
10. When $S'_3 * \mathbb{S}'_4$ acts on $F_{7,8}$, there are $\frac{8!}{3!\times(7-3)!\times(8-7)!} = 280$ equivalence classes.

### 3.2 The Set of all Mappings from X to Y

In this subsection, we discuss $F_{r,n}$. When the subgroup H acts on $F_{r,n}$, for any r-cyclic permutation $\pi_r$ in $S_r$, if $a \in F_{r,n}$ is a fixed point of $\pi_r$, then

$a(x_1) = a(\pi_r(x_1)) = a(x_2) = a(\pi_r(x_2)) = ... = a(x_r) = a(\pi_r(x_r))$.

Conversely, if the conditions above hold, it is easy to check that a is a fixed point of $\pi_r$. Thus, $|Fix(\pi_r)| = n$.

Moreover, for any $g = \pi_{s1} * \pi_{s2} \in S_r$, where $\pi_{s1}$ is an s1-cyclic permutation and $\pi_{s2}$ is an s2-cyclic permutation. If $a \in F_{r,n}$ is a fixed point of g, then

$a(x_1) = a(\pi_{s1}(x_1)) = a(x_2) = a(\pi_{s1}(x_2)) = ... = a(x_{s1}) = a(\pi_{s1}(x_{s1}))$,

$a(x_{s1+1}) = a(\pi_{s2}(x_{s1+1})) = a(x_{s1+2}) = a(\pi_{s2}(x_{s1+2})) = ... = a(x_{s1+s2}) = a(\pi_{s2}(x_{s1+s2}))$.

Conversely, if the conditions above hold, it is

easy to check that a is a fixed point of $\pi_r$ . Thus, $|Fix(\pi_{s1} * \pi_{s2})| = n^2$ .
Generally, for any $g \in H$ such that g has the form of $1^{d_1} 2^{d_2} ... r^{d_r}$ , we have that

$$|\mathrm{Fix}(g)| = n^{\sum_{i=1}^{r} d_i}.$$

In conclusion, by Lemma 2.1, when the subgroup H acts on $F_{r,n}$ , the number of equivalence classes is

$$\frac{1}{|H|}\sum_{g\in H}|\mathrm{Fix}(g)| = \frac{1}{|H|}\sum_{g\in H} n^{\sum_{i=1}^{r} d_i}.$$

Next, we use some special subgroups H acts on $F_{r,n}$.

1. When $I_r$ acts on $F_{r,n}$ , there are $n^r$ equivalence classes and each contains one element.
2. When $C_r$ acts on $F_{r,n}$ , there are

$$\frac{1}{r}\sum_{d|r}\varphi(d)\, n^{\frac{r}{d}}$$

equivalence classes.
3. When $D_{2r}$ acts on $F_{r,n}$ ($r \geq 3$), we consider the following two cases:
• If r is an odd number, then there are

$$\frac{1}{2r}\left(\sum_{d|r}\varphi(d)\, n^{\frac{r}{d}} + rn^{\frac{r+1}{2}}\right) = \frac{1}{2r}\sum_{d|r}\varphi(d)\, n^{\frac{r}{d}} + \frac{1}{2}n^{\frac{r+1}{2}}.$$

equivalence classes.
• If r is an even number, then there are

$$\frac{1}{2r}\left(\sum_{d|r}\varphi(d)\, n^{\frac{r}{d}} + \frac{r}{2}n^{\frac{r+2}{2}} + \frac{r}{2}n^{\frac{r}{2}}\right) = \frac{1}{2r}\sum_{d|r}\varphi(d)\, n^{\frac{r}{d}} + \frac{1}{4}n^{\frac{r+2}{2}} + \frac{1}{4}n^{\frac{r}{2}}.$$

equivalence classes.

4. When $S_r$ acts on $F_{r,n}$ , there are

$$\frac{1}{r!}\sum_{g\in H} n^{\sum_{i=1}^{r} d_i} = \frac{1}{r!}\sum_{d_1+2d_2+...+rd_r=r}\frac{r!}{d_1!d_2!...d_r!1^{d_1}2^{d_2}...r^{d_r}}n^{\sum_{i=1}^{r} d_i} = \frac{(n+r-1)!}{r!(n-1)!}$$

equivalence classes. For convenience, we denote $\frac{(n+r-1)!}{r!(n-1)!} = \frac{1}{r!}(t_r n^r + t_{r-1}n^{r-1} + ... + t_1 n).$

5. When $A_r$ acts on $F_{r,n}$ , where $r \geq 3$, for any $g \in A_r$ , $r - (d_1 + d_2 + ... + d_r)$ is an even integer. We consider the following two cases:
• If r is an odd integer, then $d_1 + d_2 + ... + d_r$ is an odd integer. Thus, $A_r$ has all elements g in $S_r$ that g has the form of $1^{d_1} 2^{d_2} . . . r^{d_r}$ , where $d_1 + d_2 + ... + d_r$ is an odd integer. By Lemma 2.1, the number of equivalence classes are

$$\frac{2}{r!}\sum_{g\in H} n^{\sum_{i=1}^{r} d_i} = \frac{2}{r!}\sum_{d_1+2d_2+...+rd_r=r}\frac{r!}{d_1!d_2!...d_r!1^{d_1}2^{d_2}...r^{d_r}}n^{\sum_{i=1}^{r} d_i} = \frac{2}{r!}(t_r n^r + t_{r-2}n^{r-2} + ... + t_1 n).$$

• If r is an even integer, then $d_1 + d_2 + ... + d_r$ is an even integer. Thus, $A_r$ has all elements g in $S_r$ that g has the form of $1^{d_1} 2^{d_2} . . . r^{d_r}$ , where $d_1 + d_2 + ... + d_r$ is an even integer. By Lemma 2.1, the number of equivalence classes are

$$\frac{2}{r!}\sum_{g\in H} n^{\sum_{i=1}^{r} d_i} = \frac{2}{r!}\sum_{d_1+2d_2+...+rd_r=r}\frac{r!}{d_1!d_2!...d_r!1^{d_1}2^{d_2}...r^{d_r}}n^{\sum_{i=1}^{r} d_i} = \frac{2}{r!}(t_r n^r + t_{r-2}n^{r-2} + ... + t_1 n^2).$$

6. When $C_k * C_{r-k}$ acts on $F_{r,n}$ , since $C_k$ is isomorphic to $C_k$ and $C_{r-k}$ is isomorphic to $C_{r-k}$ , $C_k$ acts on $F_{r,n}$ is equivalent to $C_k$ acts on $F_{k,n}$ and $C_{r-k}$ acts on $F_{r,n}$ is equivalent to $C_{r-k}$ acts on $F_{r-k,n}$ , respectively. Thus, there are

$$\left(\frac{1}{k}\sum_{d|k}\varphi(d)\, n^{\frac{k}{d}}\right)\left(\frac{1}{r-k}\sum_{e|(r-k)}\varphi(e)\, n^{\frac{r-k}{e}}\right)$$

equivalence classes.

7. When $D_{2k} * D_{2r-2k}$ acts on $F_{r,n}$ , where $k \geq 3$ and $r - k \geq 3$, since $D_{2k}$ is isomorphic to $D_{2k}$ and $D_{2r-2k}$ is isomorphic to $D_{2r-2k}$ , $D_{2k}$ acts on $F_{r,n}$ is equivalent to $D_{2k}$ acts on $F_{k,n}$ and $D_{2r-2k}$ acts on $F_{r,n}$ is equivalent to $D_{2r-2k}$ acts on $F_{r-k,n}$, respectively. Thus, there are a total of 4 cases because k and r − k can either be odd or even.
• If $2 \nmid k$ and $2 \nmid r - k$, then the number of equivalence classes are

$$\frac{1}{4k(r-k)}\left(\sum_{d|k}\varphi(d)\, n^{\frac{k}{d}} + kn^{\frac{k+1}{2}}\right)\left(\sum_{e|r-k}\varphi(e)\, n^{\frac{r-k}{e}} + (r-k)n^{\frac{r-k+1}{2}}\right).$$

• If $2 \nmid k$ and $2 \mid r - k$, then the number of equivalence classes are

$$\frac{1}{4k(r-k)}\left(\sum_{d|k}\varphi(d)\,n^{\frac{k}{d}}+kn^{\frac{k+1}{2}}\right)\left(\sum_{e|r-k}\varphi(e)\,n^{\frac{r-k}{e}}+\frac{r-k}{2}n^{\frac{r-k+2}{2}}+\frac{r-k}{2}n^{\frac{r-k}{2}}\right).$$

• If 2 | k and 2 ∤ r − k, then the number of equivalence classes are

$$\frac{1}{4k(r-k)}\left(\sum_{d|k}\varphi(d)\,n^{\frac{k}{d}}+\frac{k}{2}n^{\frac{k+2}{2}}+\frac{k}{2}n^{\frac{k}{2}}\right)\left(\sum_{e|r-k}\varphi(e)\,n^{\frac{r-k}{e}}+(r-k)n^{\frac{r-k+1}{2}}\right).$$

• If 2 | k and 2 | r − k, then the number of equivalence classes are

$$\frac{1}{4k(r-k)}\left(\sum_{d|k}\varphi(d)\,n^{\frac{k}{d}}+\frac{k}{2}n^{\frac{k+2}{2}}+\frac{k}{2}n^{\frac{k}{2}}\right)\left(\sum_{e|r-k}\varphi(e)\,n^{\frac{r-k}{e}}+\frac{r-k}{2}n^{\frac{r-k+2}{2}}+\frac{r-k}{2}n^{\frac{r-k}{2}}\right).$$

8. When $C_k' * D'_{2r-2k}$ acts on $F_{r,n}$, where $r-k \geq 3$, since $C'_k$ is isomorphic to $C_k$ and $D'_{2r-2k}$ is isomorphic to $D_{2r-2k}$, similarly, there are 2 cases depending on parity of r − k.•

If r is an odd number, then there are

$$\frac{1}{2k(r-k)}\left(\sum_{d|k}\varphi(d)\,n^{\frac{k}{d}}\right)\left(\sum_{e|r-k}\varphi(e)\,n^{\frac{r-k}{e}}+(r-k)n^{\frac{r-k+1}{2}}\right)$$

equivalence classes.

• If r is an even number, then there are

$$\frac{1}{2k(r-k)}\left(\sum_{d|k}\varphi(d)\,n^{\frac{k}{d}}\right)\left(\sum_{e|r-k}\varphi(e)\,n^{\frac{r-k}{e}}+\frac{r-k}{2}n^{\frac{r-k+2}{2}}+\frac{r-k}{2}n^{\frac{r-k}{2}}\right)$$

equivalence classes.

9. When $A_k' * A'_{r-k}$ acts on $F_{r,n}$, where $k \geq 3$ and $r-k \geq 3$, since $A'_k$ is isomorphic to $A_k$ and $A'_{r-k}$ is isomorphic to $A_{r-k}$, $A'_k$ acts on $F_{r,n}$ is equivalent to $A_k$ acts on $F_{k,n}$ and $A'_{r-k}$ acts on $F_{r,n}$ is equivalent to $A_{r-k}$ acts on $F_{r-k,n}$, respectively. Thus, there are also 4 cases.

• If 2 ∤ k and 2 ∤ r − k, then the number of equivalence classes are

$\frac{4}{k!(r-k)!}$ $(t_k n^k + t_{k-2} n^{k-2} + ... + t_1 n)(t_{r-k} n^{r-k} + t_{r-k-2} n^{r-k-2} + ... + t_1 n)$.

• If 2 ∤ k and 2 | r − k, then the number of equivalence classes are

$\frac{4}{k!(r-k)!}$ $(t_k n^k + t_{k-2} n^{k-2} + ... + t_1 n)(t_{r-k} n^{r-k} + t_{r-k-2} n^{r-k-2} + ... + t_2 n^2)$.

• If 2 | k and 2 ∤ r − k, then the number of equivalence classes are

$\frac{4}{k!(r-k)!}$ $(t_k n^k + t_{k-2} n^{k-2} + ... + t_2 n^2)(t_{r-k} n^{r-k} + t_{r-k-2} n^{r-k-2} + ... + t_1 n)$.

• If 2 | k and 2 | r − k, then the number of equivalence classes are

$\frac{4}{k!(r-k)!}$ $(t_k n^k + t_{k-2} n^{k-2} + ... + t_2 n^2)(t_{r-k} n^{r-k} + t_{r-k-2} n^{r-k-2} + ... + t_2 n^2)$.

10. When $S_k' * S'_{r-k}$ acts on $F_{r,n}$, since $S'_k$ is isomorphic to $S_k$ and $S'_{r-k}$ is isomorphic to $S_{r-k}$, $S'_k$ acts on $F_{r,n}$ is equivalent to $S_k$ acts on $F_{k,n}$ and $S'_{r-k}$ acts on $F_{r,n}$ is equivalent to $S_{r-k}$ acts on $F_{r-k,n}$, respectively. Therefore, there are

$\frac{1}{k!(r-k)!}$ $(t_k n^k + t_{k-1} n^{k-1} + ... + t_1 n)(t_{r-k} n^{r-k} + t_{r-k-1} n^{r-k-1} + ... + t_1 n)$ equivalence classes.

**Example 3.2.** If r = 7, k = 3 and n = 8, then we have that

1. When $I_7$ acts on $F_{7,8}$, there are $8^7 = 2097152$ equivalence classes.

2. When $C_7$ acts on $F_{7,8}$, there are

$\frac{1}{7}$ × (φ(1) × $8^{\frac{7}{1}}+\varphi(7)\times 8^{\frac{7}{7}}\Big)=\frac{1}{7}\times(1\times 8^7+6\times 8)=299600$

equivalence classes.

3. When $D_{14}$ acts on $F_{7,8}$, there are

$\frac{1}{14}$ × (φ(1) ×

equivalence classes

4. When $S_7$ acts on $F_{7,8}$, there are

$\frac{(8+7-1)!}{7!\times(8-1)!}=3432$

equivalence classes.

5. When $A_7$ acts on $F_{7,8}$, there are

$\frac{2}{7!}$ × (1 × $8^7$ + 175 × $8^5$ + 1624 × $8^3$ + 720 × $8^1$) = 3440

equivalence classes.

6. When C3 * C4 acts on F7 ,8 , there are

$\frac{1}{12} \times (\varphi(1) \times 8^{\frac{3}{1}} + \varphi(3) \times 8^{\frac{3}{3}}) \times (\varphi(1) \times 8^{\frac{4}{1}} + \varphi(2) \times 8^{\frac{4}{2}} + \varphi(4) \times 8^{\frac{4}{4}}) = \frac{1}{12} \times (1 \times 83 + 2 \times 8) \times (1 \times 84 + 1 \times 82 + 2 \times 8)$

=183744

equivalence classes.

7. When D6′ * D8 acts on F7 ,8 , there are

$\frac{1}{48} \times (\varphi(1) \times 8^{\frac{3}{1}} + \varphi(3) \times 8^{\frac{3}{3}} + 3 \times 8^{\frac{3+1}{2}}) \times (\varphi(1) \times 8^{\frac{4}{1}} + \varphi(2) \times 8^{\frac{4}{2}} + \varphi(4) \times 8^{\frac{4}{4}} + \frac{4}{2} \times 8^{\frac{4+2}{2}} + \frac{4}{2} \times 8^{\frac{4}{2}})$

$= \frac{1}{48} \times (1 \times 83 + 2 \times 81 + 3 \times 82 ) \times (1 \times 84 + 1 \times 82 + 2 \times 8 + 2 \times 83 + 2 \times 82 )$

=79920

equivalence classes.

8. When C3 * D8 acts on F7 ,8 , there are

$\frac{1}{24} \times (\varphi(1) \times 8^{\frac{3}{1}} + \varphi(3) \times 8^{\frac{3}{3}}) \times (\varphi(1) \times 8^{\frac{4}{1}} + \varphi(2) \times 8^{\frac{4}{2}} + \varphi(4) \times 8^{\frac{4}{4}} + \frac{4}{2} \times 8^{\frac{4+2}{2}} + \frac{4}{2} \times 8^{\frac{4}{2}}) = \frac{1}{24} \times (1 \times 83 + 2 \times 8) \times (1 \times 84 + 1 \times 82 + 2 \times 8 + 2 \times 83 + 2 \times 82 )$

=177216

equivalence classes.

9. When A3 * A4 acts on F7 ,8 , there are

$\frac{4}{3! \times 4!} \times (1 \times 83 + 2 \times 8) \times (1 \times 84 + 11 \times 82 )$

= 70400

equivalence classes.

10. When S3 * S4 acts on F7 ,8 , there are

$\frac{1}{3! \times 4!} \times (1 \times 83 + 3 \times 82 + 2 \times 8) \times (1 \times 84 + 6 \times 83 + 11 \times 82 + 6 \times 81 ) = \frac{10!}{3! \times 7!} \times \frac{11!}{4! \times 7!} =$ 39600

equivalence classes.

**Example 3.3.** If r = 8, k = 3 and n = 3, then we have that

1. When I8 acts on F8 ,3 , there are 38 = 6561 equivalence classes.

2. When C8 acts on F8 ,3 , there are

$$\frac{1}{8} \times (\varphi(1) \times 3^{\frac{8}{1}} + \varphi(2) \times 3^{\frac{8}{2}} + \varphi(4) \times 3^{\frac{8}{4}} + \varphi(8) \times 3^{\frac{8}{8}}) = \frac{1}{8} \times (1 \times 3^8 + 1 \times 3^4 + 2 \times 3^2 + 4 \times 3) = 834$$

equivalence classes.

3. When D16 acts on F8 ,3 , there are

$\frac{1}{16} \times (\varphi(1) \times 3^{\frac{8}{1}} + \varphi(2) \times 3^{\frac{8}{2}} + \varphi(4) \times 3^{\frac{8}{4}} + \varphi(8) \times 3^{\frac{8}{8}} + \frac{8}{2} \times 3^{\frac{8+2}{2}} + \frac{8}{2} \times 3^{\frac{8}{2}}) = \frac{1}{16} \times (1 \times 38 + 1 \times 34 + 2 \times 32 + 4 \times 31 + 4 \times 35 + 4 \times 34 )$

=498

equivalence classes.

4. When S8 acts on F8 ,3 , there are

$\frac{(3 + 8 - 1)!}{8! \times (3 - 1)!} = 45$

equivalence classes.

5. When A8 acts on F8 ,3 , there are

$\frac{2}{8!} \times (1 \times 38 + 322 \times 36 + 6769 \times 34 + 13068 \times 32 ) = 45$

equivalence classes.

6. When C3 * C5 acts on F8 ,3 , there are

$$\frac{1}{15} \times (\varphi(1) \times 3^{\frac{3}{1}} + \varphi(3) \times 3^{\frac{3}{3}}) \times (\varphi(1) \times 3^{\frac{5}{1}} + \varphi(5) \times 3^{\frac{5}{5}}) = \frac{1}{15} \times (1 \times 3^3 + 2 \times 3^1) \times (1 \times 3^5 + 4 \times 3^1) = 561$$

equivalence classes.

7. When D6′ * D′10 acts on F8 ,3 , there are

$$\frac{1}{60} \times (\varphi(1) \times 3^{\frac{3}{1}} + \varphi(3) \times 3^{\frac{3}{3}} + 3 \times 3^{\frac{3+1}{2}}) \times (\varphi(1) \times 3^{\frac{5}{1}} + \varphi(5) \times 3^{\frac{5}{5}} + 5 \times 3^{\frac{5+1}{2}})$$

$$=\frac{1}{60}\times(1\times 3^3+2\times 3^1+3\times 3^2)\times(1\times 3^5+4\times 3+5\times 3^3)$$

=390
equivalence classes.

8. When C3 ∗ D10 acts on F8 ,3 , there are

$$\frac{1}{30}\times\left(\varphi(1)\times 3^{\frac{3}{1}}+\varphi(3)\times 3^{\frac{3}{3}}\right)\times\left(\varphi(1)\times 3^{\frac{5}{1}}+\varphi(5)\times 3^{\frac{5}{5}}+5\times 3^{\frac{5+1}{2}}\right)$$
$$=\frac{1}{30}\times(1\times 3^3+2\times 3)\times(1\times 3^5+4\times 3+5\times 3^3)$$

=429
equivalence classes.

9. When A3 ∗ A5 acts on F8 ,3 , there are

$$\frac{4}{3!\times 5!}\times(1\times 3^3+2\times 3^1)\times(1\times 3^5+35\times 3^3+24\times 3^1)=231$$

equivalence classes.

10. When S3 ∗ S5 acts on F8 ,3 , there are

$$\frac{(3+3-1)!}{3!\times(3-1)!}\times\frac{(3+5-1)!}{5!\times(3-1)!}=210$$

equivalence classes.

### 3.3 The Set of Some Special Mappings from X to Y

We can think of a special kind of mappings from X to Y with m1 y1 s, m2 y2 s,. . . , mn yns, so that $\sum_{i=1}^{n}$ mi = r , mi ∈ N, i = 1, 2, ...,n and r ≥ n. The set of such permutations is Zm 1 ,...,mn . We can see that Zm 1 ,...,mn has

$$\frac{(m_1+m_2+\ldots+m_n)!}{m_1!m_2!\cdots m_n!}$$

elements. In this subsection, we discuss the case of the subgroup H of Sr acts on Zm 1 ,...,mn . For any g ∈ H such that g has the form of 1d1 2d2 . . . rdr in Sr . We denote cij as the number of i-cyclic permutation for yj , i = 1, 2, ...,r and j = 1, 2, ...,n. If a is a fixed point if g, then the following Diophantine equation

$$\begin{cases} c_{11}+2c_{21}+\ldots+rc_{r1}=m_1\\ c_{12}+2c_{22}+\ldots+rc_{r2}=m_2\\ \quad\cdots\\ c_{1n}+2c_{2n}+\ldots+rc_{rn}=m_n\\ c_{11}+c_{12}+\ldots+c_{1n}=d_1\\ c_{21}+c_{22}+\ldots+c_{2n}=d_2\\ \quad\cdots\\ c_{r1}+c_{r2}+\ldots+c_{rn}=d_r \end{cases}$$

has an integer solution. Conversely, it is easy to check that if the equation above has an integer solution, then g has a fixed point and every solution means

$$\frac{d_1!}{c_{11}!\cdots c_{1n}!}\frac{d_2!}{c_{21}!\cdots c_{2n}!}\cdots\frac{d_r!}{c_{r1}!\cdots c_{rn}!}$$

fixed points of g. If this equation does not have an integer solution, g does not have a fixed point. According to Lemma 2.1, when the subgroup H of Sr acts on Zm 1 ,...,mn , the number of equivalence classes is

$$\frac{1}{|H|}\sum_{g\in H}|\mathrm{Fix}(g)|.$$

Next, we use some special subgroups H acts on Zm 1 ,...,mn .

1. When Ir acts on Zm 1 ,...,mn there are $\frac{(m_1+m_2+\ldots+m_n)!}{m_1!m_2!\ \ m_n!}$ ⋯ equivalence classes and each contains one element.

2. When Cr acts on Zm 1 ,...,mn , for any d|r , Cr has φ(d) elements of the form $d^{\frac{r}{d}}$. For any $g\in C_r$ has the form of $d^{\frac{r}{d}}$, the corresponding Diophantine equation is

$$\begin{cases} dc_{d1}=m_1\\ dc_{d2}=m_2\\ \quad\cdots\\ dc_{dn}=m_n\\ c_{d1}+c_{d2}+\ldots+c_{dn}=\frac{r}{d} \end{cases}$$

If mi |r , i = 1, 2,...,n, then the Diophantine equation above has a unique integer solution and g has

$$\frac{(\frac{r}{d})!}{(\frac{m_1}{d})!\ldots(\frac{m_n}{d})!}$$

fixed points. Otherwise, g has no fixed points. Thus, by Lemma 2.1, we obtain the number of equivalence classes.

3. When D2r acts on Zm 1 ,...,mn , for any d|r ,

- If $r$ is an odd integer, then there exists $\varphi(d)$ elements of the form $d^{\frac{r}{d}}$ and $r$ elements of the form

12 $\frac{r-1}{2}$ in Cr . The case of the form $d^{\frac{r}{d}}$ is discussed previously. If $g$ has the form of $12^{\frac{r-1}{2}}$, then the corresponding Diophantine equation is

$$\begin{cases} c_{11}+2c_{21}=m_1\\ c_{12}+2c_{22}=m_2\\ \quad\cdots\\ c_{1n}+2c_{2n}=m_n\\ c_{11}+c_{12}+\ldots+c_{1n}=1\\ c_{21}+c_{22}+\ldots+c_{2n}=\frac{r-1}{2} \end{cases}$$

If the Diophantine equation above has an integer solution, then g has

$$\frac{(\frac{r-1}{2})!}{c_{21}!\cdots c_{2n}!}$$

fixed points. Otherwise, g has no fixed points.

Thus, by Lemma 2.1, we obtain the number of equivalence classes.

• If r is an even integer, then there exists φ(d) elements of the form $d^{\frac{r}{d}}$ , $\frac{r}{2}$ elements of the form 122
$\frac{r-2}{2}$, and $\frac{r}{2}$ elements of the form $2^{\frac{r}{2}}$ in $C_r$. The case of the form $d^{\frac{r}{d}}$ and $2^{\frac{r}{2}}$ are discussed previously. If g has the form of 122
$\frac{r-2}{2}$ , then the corresponding Diophantine equation is

$$\begin{cases} c_{11} + 2c_{21} = m_1 \\ c_{12} + 2c_{22} = m_2 \\ \cdots \\ c_{1n} + 2c_{2n} = m_n \\ c_{11} + c_{12} + \ldots + c_{1n} = 2 \\ c_{21} + c_{22} + \ldots + c_{2n} = \frac{r-2}{2} \end{cases}$$

If the Diophantine equation above has an integer solution, then g has

$$\frac{2!}{c_{11}! \cdots c_{1n}!} \frac{(\frac{r-2}{2})!}{c_{21}! \cdots c_{2n}!}$$

fixed points. Otherwise, g has no fixed points. Thus, by Lemma 2.1, we obtain the number of equivalence classes.

4. When Sr acts on Zm 1 ,...,mn , for any a ∈ Zm 1 ,...,mn , there exists m1 !m2 !...mn! elements in Sr such that a can be a fixed point of them. By Lemma 2.1, the number of equivalence classes is

$$\frac{1}{|\mathbb{S}_r|} \sum_{g \in \mathbb{S}_r} |\mathrm{Fix}(g)| = \frac{1}{r!} \frac{r!}{m_1!...m_n!} (m_1!...m_n!) = 1.$$

In other words, when Sr acts on Zm 1 ,...,mn , it only creates 1 equivalence class, and every element is considered equivalent.

5. When Ar acts on Zm 1 ,...,mn , for any a ∈ Zm 1 ,...,mn , there exists m 1 !m22!...mn! elements in Ar such that a can be a fixed point of them. By Lemma 2.1, the number of equivalence classes is

$$\frac{1}{|A_r|} \sum_{g \in A_r} |\mathrm{Fix}(g)| = \frac{2}{r!} \frac{r!}{m_1!...m_n!} \frac{m_1!...m_n!}{2} = 1.$$

In other words, when Ar acts on Zm 1 ,...,mn , it only creates 1 equivalence class, and every element is considered equivalent.

For any subgroup H1 in Sk and subgroup H2 in Sr−k, when an inner direct product group H1 ∗ H2 acts on Zm 1 ,...,mn , if g1 ∈ H1 has the form 1d1 2d2 ...rdr in Sr and g2 ∈ H2 has the form 1e1 2e2 ...rer in Sr , then g1 ∗ g2 has the form of 1d1 +e1 −r2d2 +e2 ...rdr +er . Thus, if we obtain the form of all elements in H1 ∗ H2 , then we can solve the corresponding Diophantine equations and calculate the number of equivalence classes by Lemma 2.1.

**Example 3.4.** If r = 8, k = 3, m1 = 3, m2 = 3, and m3 = 2, then we have that 1. When I8 acts on Z3 ,3 ,2 , there are

$$\frac{8!}{3! \times 3! \times 2!} = 560$$

equivalence classes

2. When C8 acts on Z3 ,3 ,2 , we consider the following 4 cases:

(a) C8 has an element of the form 18 , and this element has $\frac{8!}{3!3!2!}$ = 560 fixed points. (b) C8 has an element of the form 24 , but there are no fixed points.

(c) C8 has 2 elements of the form 42 , but there are no fixed points.

(d) C8 has 4 elements of the form 81 , but there are no fixed points. Thus, there are

$$\frac{1}{8} \times 560 = 70$$

equivalence classes.

3. When D16 acts on Z3 ,3 ,2 , we consider the following 5 cases:

(a) D16 has an element of the form 18 , and this element has $\frac{8!}{3!3!2!}$ = 560 fixed points. (b) D16 has 5 elements of the form 24 , but there are no fixed points.

(c) D16 has 2 elements of the form 42 , but there are no fixed point

(d) D16 has 4 elements of the form 81 , but there are no fixed points.

(e) D16 has 4 elements of the form 1223 , and each element has $\frac{2!}{1!1!0!} \times \frac{3!}{1!1!1!}$ = 12 fixed points. Thus, there are

$$\frac{1}{16} \times (1 \times 560 + 4 \times 12) = 38$$

equivalence classes.

4. When S8 acts on Z3 ,3 ,2 , there is 1 equivalence classes.

5. When A8 acts on Z3 ,3 ,2 , there is 1 equivalence classes.

6. When C 3 ∗ C 5 acts on Z3 ,3 ,2 , we consider the following 4 cases:

(a) C3 ∗ C5 has an element of the form 18 , and this element has $\frac{8!}{3!3!2!}$ = 560 fixed points.

(b) C3 ∗ C5 has 4 elements of the form 1351 , but there are no fixed points.

(c) C3 ∗ C5 has 2 elements of the form 1531 , and each element has $\frac{5!}{0!3!2!} \times \frac{1!}{1!0!0!} + \frac{5!}{3!0!2!} \times \frac{1!}{0!1!0!}$ = 20 fixed points.

(d) C3 ∗ C5 has 8 elements of the form 3151 , but there are no fixed points.

Thus, there are

$\frac{1}{15} \times (1 \times 560 + 2 \times 20) = 40$

equivalence classes.

7. When $D6 * D10$ acts on $Z3, 3, 2$, we consider the following 9 cases:

(a) $D6 * D'10$ has an element of the form $1^8$, and this element has $\frac{8!}{3!3!2!} = 560$ fixed points.

(b) $D6 * D'10$ has 3 element of the form $1^6 2^1$, and each element has $\frac{6!}{3!3!0!} \times \frac{1!}{0!0!1!} + \frac{6!}{3!1!2!} \times \frac{1!}{0!1!0!} + \frac{6!}{1!3!2!} \times \frac{1!}{1!0!0!} = 140$ fixed points.

(c) $D6 * D'10$ has 2 elements of the form $1^5 3^1$, and each element has $\frac{5!}{0!3!2!} \times \frac{1!}{1!0!0!} + \frac{5!}{3!0!2!} \times \frac{1!}{0!1!0!} = 20$ fixed points.

(d) $D6 * D'10$ has 5 elements of the form $1^4 2^2$, and each element has $\frac{4!}{3!1!0!} \times \frac{2!}{0!1!1!} + \frac{4!}{1!3!0!} \times \frac{2!}{1!0!1!} + \frac{4!}{1!1!2!} \times \frac{2!}{1!1!0!} = 40$ fixed points.

(e) $D6 * D'10$ has 15 elements of the form $1^2 2^3$, and each element has $\frac{2!}{1!1!0!} \times \frac{3!}{1!1!1!} = 12$ fixed points.

(f) $D6 * D'10$ has 10 elements of the form $1^1 2^2 3^1$, and each element has $\frac{1!}{1!0!0!} \times \frac{2!}{1!0!1!} \times \frac{1!}{0!1!0!} + \frac{1!}{0!1!0!} \times \frac{2!}{0!1!1!} \times \frac{1!}{1!0!0!} = 4$ fixed points.

(g) $D6 * D'10$ has 4 elements of the form $1^3 5^1$, but there are no fixed points.

(h) $D6 * D'10$ has 12 elements of the form $1^1 2^1 5^1$, but there are no fixed points.

(i) $D6 * D'10$ has 8 elements of the form $3^1 5^1$, but there are no fixed points.

Thus, there are

$\frac{1}{60} \times (1 \times 560 + 3 \times 140 + 2 \times 20 + 5 \times 40 + 15 \times 12 + 10 \times 4) = 24$

equivalence classes.

8. When $C3 * D10$ acts on $Z3, 3, 2$, we consider the following 6 cases:

(a) $C3 * D'10$ has an element of the form $1^8$, and this element has $\frac{8!}{3!3!2!} = 560$ fixed points.

(b) $C3 * D'10$ has 2 elements of the form $1^5 3^1$, and each element has $\frac{5!}{0!3!2!} \times \frac{1!}{1!0!0!} + \frac{5!}{3!0!2!} \times \frac{1!}{0!1!0!} = 20$ fixed points.

(c) $C3 * D'10$ has 4 elements of the form $1^3 5^1$, but there are no fixed points.

(d) $C3 * D'10$ has 8 elements of the form $3^1 5^1$, but there are no fixed points.

(e) $C3 * D'10$ has 5 elements of the form $1^4 2^2$, and each element has $\frac{4!}{3!1!0!} \times \frac{2!}{0!1!1!} + \frac{4!}{1!3!0!} \times \frac{2!}{1!0!1!} + \frac{4!}{1!1!2!} \times \frac{2!}{1!1!0!} = 40$ fixed points.

(f) $C3 * D'10$ has 10 elements of the form $1^1 2^2 3^1$, and each element has $\frac{1!}{1!0!0!} \times \frac{2!}{1!0!1!} \times \frac{1!}{0!1!0!} + \frac{1!}{0!1!0!} \times \frac{2!}{0!1!1!} \times \frac{1!}{1!0!0!} = 4$ fixed points.

Thus, there are

$\frac{1}{30} \times (1 \times 560 + 2 \times 20 + 5 \times 40 + 10 \times 4) = 28$

equivalence classes

9. When $A3 * A5$ acts on $Z3, 3, 2$, we consider the following 7 cases:

(a) $A3 * A5$ has an element of the form $1^8$, and this element has $\frac{8!}{3!3!2!} = 560$ fixed points.

(b) $A3 * A5$ has 22 elements of the form $1^5 3^1$, and each element has $\frac{5!}{0!3!2!} \times \frac{1!}{1!0!0!} + \frac{5!}{3!0!2!} \times \frac{1!}{0!1!0!} = 20$ fixed points.

(c) $A3 * A5$ has 40 elements of the form $1^2 3^2$, and each element has $\frac{2!}{0!0!2!} \times \frac{2!}{1!1!0!} = 2$ fixed points.

(d) $A3 * A5$ has 15 elements of the form $1^4 2^2$, and each element has $\frac{4!}{3!1!0!} \times \frac{2!}{0!1!1!} + \frac{4!}{1!3!0!} \times \frac{2!}{1!0!1!} + \frac{4!}{1!1!2!} \times \frac{2!}{1!1!0!} = 40$ fixed points.

(e) $A3 * A5$ has 30 elements of the form $1^1 2^2 3^1$, and each element has $\frac{1!}{1!0!0!} \times \frac{2!}{1!0!1!} \times \frac{1!}{0!1!0!} + \frac{1!}{0!1!0!} \times \frac{2!}{0!1!1!} \times \frac{1!}{1!0!0!} = 4$ fixed points.

(f) $A3 * A5$ has 24 elements of the form $1^3 5^1$, but there are no fixed points. (g) $A3 * A5$ has 48 elements of the form $3^1 5^1$, but there are no fixed points.

Thus, there are

$\frac{4}{3! \times 5!} \times (1 \times 560 + 22 \times 20 + 40 \times 2 + 15 \times 40 + 30 \times 4) = 10$

equivalence classes.

10. When $S3 * S5$ acts on $Z3, 3, 2$, we consider the following 15 cases:

(a) $S3 * S5$ has an element of the form $1^8$, and this element has $\frac{8!}{3!3!2!} = 560$ fixed points.

(b) $S3 * S5$ has 13 element of the form $1^6 2^1$, and each element has $\frac{6!}{3!3!0!} \times \frac{1!}{0!0!1!} + \frac{6!}{3!1!2!} \times \frac{1!}{0!1!0!}$

$+ \frac{6!}{1!3!2!} \times \frac{1!}{1!0!0!} = 140$ fixed points.

(c) S3 * S5 has 22 elements of the form 1531 , and each element has $\frac{5!}{0!3!2!} \times \frac{1!}{1!0!0!} + \frac{5!}{3!0!2!} \times \frac{1!}{0!1!0!}$ = 20 fixed points.

(d) S3 * S5 has 45 elements of the form 1422 , and each element has $\frac{4!}{3!1!0!} \times \frac{2!}{0!1!1!} + \frac{4!}{1!3!0!} \times \frac{2!}{1!0!1!} + \frac{4!}{1!1!2!} \times \frac{2!}{1!1!0!} = 40$ fixed points.

(e) S3 * S5 has 100 elements of the form 132131 , and each element has $\frac{3!}{3!0!0!} \times \frac{1!}{0!0!1!} \times \frac{1!}{0!1!0!} + \frac{3!}{0!3!0!} \times \frac{1!}{0!0!1!} \times \frac{1!}{1!0!0!} + \frac{3!}{0!1!2!} \times \frac{1!}{0!1!0!} \times \frac{1!}{1!0!0!} + \frac{3!}{1!0!2!} \times \frac{1!}{1!0!0!} \times \frac{1!}{0!1!0!} = 8$ fixed points.

(f) S3 * S5 has 40 elements of the form 1232 , and each element has $\frac{2!}{0!0!2!} \times \frac{2!}{1!1!0!} = 2$ fixed points. (g) S3 * S5 has 45 elements of the form 1223 , and each element has $\frac{2!}{1!1!0!} \times \frac{3!}{1!1!1!} = 12$ fixed points.

(h) S3 * S5 has 90 elements of the form 112231 , and each element has $\frac{1!}{1!0!0!} \times \frac{2!}{1!0!1!} \times \frac{1!}{0!1!0!} + \frac{1!}{0!1!0!} \times \frac{2!}{0!1!1!} \times \frac{1!}{1!0!0!} = 4$ fixed points.

(i) S3 * S5 has 30 elements of the form 1441 , but there are no fixed points.

(j) §3, * §5, has 90 elements of the form 122141 , but there are no fixed points. (k) §3, * §5, has 60 elements of the form 113141 , but there are no fixed points.

(l) §3, * §5, has 40 elements of the form 2132 , and each element has $\frac{1!}{0!0!1!} \times \frac{2!}{1!1!0!} = 2$ fixed points.

(m) §3, * §5, has 24 elements of the form 1351 , but there are no fixed points. (n) §3, * §5, has 72 elements of the form 112151 , but there are no fixed points. (o) §3, * §5, has 48 elements of the form 3151 , but there are no fixed points. Thus, there are

$\frac{1}{3! \times 5!} \times (1 \times 560 + 13 \times 140 + 22 \times 20 + 45 \times 40 + 100 \times 8 + 40 \times 2 + 45 \times 12 + 90 \times 4 + 40 \times 2) = 9$

equivalence classes.

## 4. Applications

In this section, we give some application of this topic. For example,

1. How many ways are there to arrange 9 people in a circle? In this case it's C9 acting on ᒍᒐ9 ,9 . We need to calculate the number of equivalence classes when C9 acts on 9 ,9 . There are

$$\frac{9!}{9 \times (9-9)!} = 8! = 40320$$

equivalence classes, thus there are 40320 ways to arrange 9 people in a circle.

2. How many necklaces can be made from 3 red beads, 3 yellow beads, and 3 green beads? In this case it's C9 acting on 33 ,3 ,3 . We need to calculate the number of equivalence classes when C9 acts on 33 ,3 ,3 . We consider the following 3 cases:

(a) C9 has an element of the form 19 , and this element has $\frac{9!}{3!3!3!} = 1680$ fixed points. (b) C9 has 2 elements of the form 33 , and each element has $\frac{3!}{1!1!1!} = 6$ fixed points.

(c) C9 has 6 elements of the form 91 , but there are no fixed points. Thus, there are

$$\frac{1}{9} \times (1 \times 1680 + 2 \times 6) = 188$$

equivalence classes. In other words, there are 188 different kinds of necklaces which can be made from

3 red beads, 3 yellow beads, and 3 green beads.

3. How many bracelets can be made from 3 red beads, 3 yellow beads, and 3 green beads? In this case it's D18 acting on 33 ,3 ,3 . We need to calculate the number of equivalence classes when D18 acts on 33 ,3 ,3 . We consider the following 4 cases:

(a) D18 has an element of the form 19 , and this element has $\frac{9!}{3!3!3!} = 1680$ fixed points. (b) D18 has 2 elements of the form 33 , and each element has $\frac{3!}{1!1!1!} = 6$ fixed points.

(c) D18 has 6 elements of the form 91 , but there are no fixed points.

(d) D18 has 9 elements of the form 1124 , but there are no fixed points. Thus, there are

$\frac{1}{18} \times (1 \times 1680 + 2 \times 6) = 94$

equivalence classes. In other words, there are 94 different kinds of bracelets which can be made from

3 red beads, 3 yellow beads, and 3 green beads.

4. How many bead sequences can be made from 3 red beads, 3 yellow beads, and 3 green beads, and the first 4 beads and the last 5 beads are each considered as necklaces, respectively? In this case it's

C4 × C5 acting on Z3 ,3 ,3 . We need to calculate the number of equivalence classes when C4 × C5 acts on Z3 ,3 ,3 . We consider the following 6 cases:

(a) C4 × C5 has an element of the form 19 , and this element has $\frac{9!}{3!3!3!} = 1680$ fixed points.

(b) C4 × C5 has 2 elements of the form 1541 , but there are no fixed points.

(c) C4 × C5 has an element of the form 1522 , and each element has $\frac{5!}{3!1!1!} \times \frac{2!}{0!1!1!} \times 3 = 120$ fixed points.

(d) C4 × C5 has 4 elements of the form 1451 , but there are no fixed points. (e) C4 × C5 has 4 elements of the form 2251 , but there are no fixed points. (f) C4 × C5 has 8 elements of the form 4151 , but there are no fixed points.

Thus, there are

$$\frac{1}{20} \times (1 \times 1680 + 1 \times 120) = 90$$

equivalence classes. In other words, there are 90 different kinds of necklaces which can be made from 3 red beads, 3 yellow beads, and 3 green beads and the first 4 beads and the last 5 beads are each considered as necklaces, respectively.

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